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# A GEOMETRIC REPRESENTATION OF THE QUASI-TRIGONOMETRIC FUNCTIONS

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F. MAX STEIN, Colorado State University

**1. Introduction.** In 1962 Strand and Stein [3] introduced the notion of *quasi-trigonometric functions* by the use of an oblique coordinate system, shown later in (11) and Figure 4. When the angle $\lambda$ between the coordinate axis is $\pi/2$, the usual trigonometric functions are obtained. For $0<\lambda\leqq\pi/2$, *quasi-elliptic trigonometry* results, and if $\lambda=0$, *quasi-parabolic trigonometry* is obtained. When $\lambda$ assumes imaginary values we get *quasi-hyperbolic trigonometry*. Thus as $\lambda$ varies, a tenuous connection between the trigonometric and hyperbolic functions is obtained. Incidentally Carter and Stein [1] have shown these functions can be defined as solutions of certain nonlinear differential equations.

In this paper we present a three dimensional geometric representation for defining the quasi-trigonometric functions. These functions can be defined in a plane that intersects the hyperboloid of revolution

$$x^2 - y^2 + z^2 = 1. \tag{1}$$

The intersection of the plane $y=0$ and the hyperboloid of one sheet (1) is a unit circle. We define the plane $Q$ as a plane that rotates about the $z$-axis and is initially the plane $y=0$. The intersection of $Q$ and the surface (1) will vary from a unit circle, to an ellipse, to a degenerate parabola, to a general hyperbola, and finally to an equilateral hyperbola.

If the angle between the $y$-axis and the rotating plane $Q$ is called $\beta$, then $\beta$ will determine which quasi-trigonometric functions are obtained. (See Figure 1.) In particular, when $\beta=\pi/2$ the intersection of $Q$ and the surface (1) is a unit circle. For $\pi/2<\beta<\pi/4$ the intersection of $Q$ and the surface (1) is an ellipse. When $\beta=\pi/4$ the intersection is a pair of parallel lines, i.e., a degenerate parabola, since the hyperbola of one sheet is a ruled surface. For $0\leqq\beta<\pi/4$ the intersection of $Q$ and the surface (1) is a hyperbola; an equilateral hyperbola results when $\beta=0$.

**2. Quasi-elliptic functions.** In order to define the *quasi-elliptic functions* we first define a $w$-axis as the intersection of the planes $z=0$ and $Q$. Since the intersection of the surface (1) and $Q$ is an ellipse in the $Q$ plane, the ellipse thus has the general form

$$\frac{w^2}{b^2} + z^2 = 1, \tag{2}$$

when $\pi/4<\beta\leqq\pi/2$, with the semimajor and semiminor axes being $b$ ($b\geqq 1$) and 1 respectively.

Next we define the $x'$- and $y'$-axes in the rotating plane $Q$ by the equations

$$z = -w\tan\lambda/2 \quad\text{and}\quad z = w\tan\lambda/2, \tag{3}$$

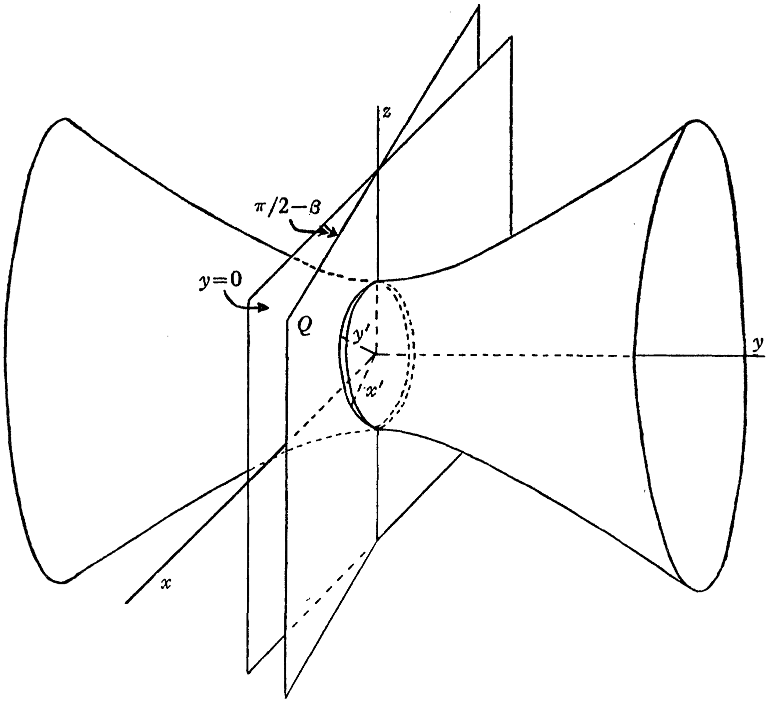


FIG. 1.

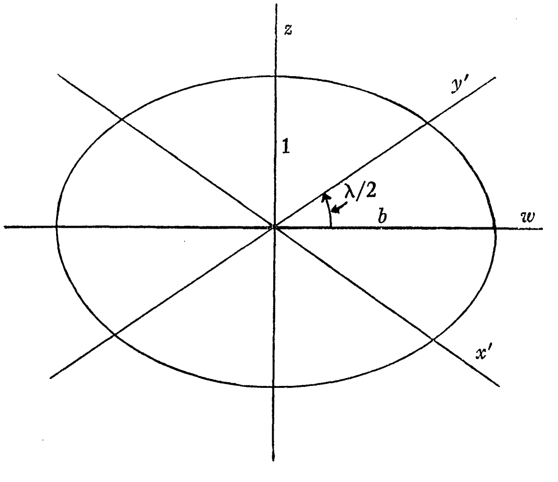


FIG. 2.

respectively, where

$$\tan \lambda/2 = 1/b. \tag{4}$$

That is, the angle between the $x'$- and $y'$-axes is $\lambda$, and this angle is bisected by the $w$-axis. (See Figure 2.)

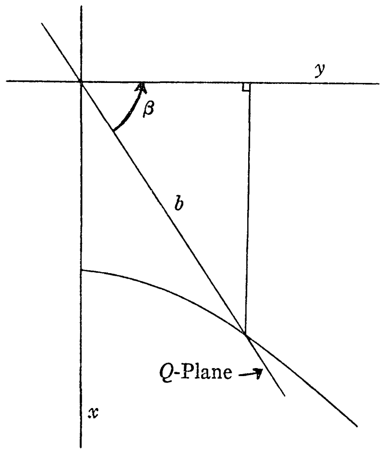


FIG. 3.

Figure 3 shows the intersection in the $z=0$ plane of a portion of a trace of the surface (1) and the trace of $Q$ as viewed from along the positive $z$-axis. From (1) and Figure 3 we see that the equation of the rotating plane $Q$ is

$$y = x \cot \beta \tag{5}$$

and also that

$$b = \sqrt{(x^2 + y^2)} \tag{6}$$

(in the $z=0$ plane, of course). Thus the point of intersection of the surface (1), the plane $z=0$, and the plane $Q$ is

$$x = \frac{1}{\sqrt{(1 - \cot^2 \beta)}}, \quad y = \frac{\cot \beta}{\sqrt{(1 - \cot^2 \beta)}}, \quad z = 0, \tag{7}$$

for any $\beta$ such that $\pi/4 < \beta \leqq \pi/2$.

From (6) and (7) we see that

$$b = \sqrt{\frac{1}{1 - \cot^2 \beta} + \frac{\cot^2 \beta}{1 - \cot^2 \beta}} \tag{8}$$

which reduces to

$$b = \sqrt{-\sec 2\beta}. \tag{9}$$

From (4) and (9) we see that the relationship between $\lambda$ and $\beta$ is

$$\tan \lambda/2 = \sqrt{-\cos 2\beta}. \tag{10}$$

In [3] the quasi-elliptic functions were defined as

$$\operatorname{nis} \theta = \frac{y'}{r} = \frac{\sin \theta}{\sin \lambda}, \qquad \operatorname{conis} \theta = \frac{x'}{r} = \frac{\sin (\lambda - \theta)}{\sin \lambda},$$

$$\text{(11)} \qquad \operatorname{nat}\theta = \frac{y'}{x'} = \frac{\operatorname{nis}\theta}{\operatorname{conis}\theta}, \qquad \operatorname{conat}\theta = \frac{x'}{y'} = \frac{1}{\operatorname{nat}\theta},$$

$$\operatorname{ces}\theta = \frac{r}{x'} = \frac{1}{\operatorname{conis}\theta}, \qquad \operatorname{coces}\theta = \frac{r}{y'} = \frac{1}{\operatorname{nis}\theta}.$$

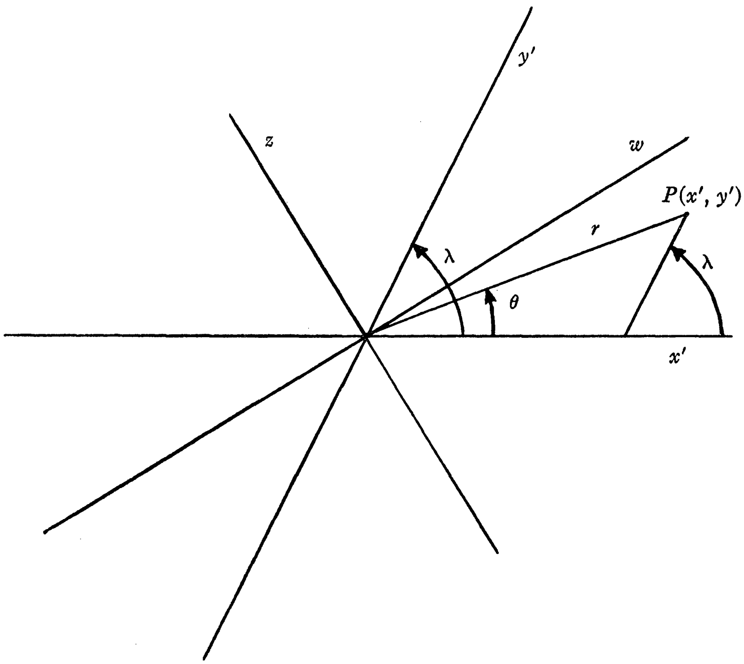


FIG. 4.

(See Figure 4 which is obtained from Figure 2 by rotating the figure to make the $x'$-axis horizontal.) But the functions in (11) are the same as those obtained by considering a point $P$ on the ellipse, Figure 2 and Figure 4, and defining functions analogous to the circular functions defined when $Q$ coincides with the $y=0$ plane. Thus as $Q$ rotates about the $z$-axis with $\pi/4<\beta<\pi/2$, we are able to define the quasi-elliptic functions, as defined by Strand and Stein [3], in terms of the intersection of $Q$ and the surface (1); and (10) shows that $\lambda$, the angle between the $x'$- and $y'$-axes, is determined by $\beta$.

**3. Quasi-hyperbolic functions.** We now consider values of $\beta$ for $0\leqq\beta<\pi/4$. In Section 1 it was observed that the intersection of $Q$ and the surface (1) is a general hyperbola; the equation of this hyperbola in the $zw$-plane, or $Q$ plane, is

$$\text{(12)} \qquad z^2 - \frac{w^2}{a^2} = 1$$

with 1 as the semitransverse and $a$ ($a\geqq 1$) as the semiconjugate axis. Note that when $\beta=0$ the $w$-axis and the $y$-axis coincide, and the $x=0$ plane coincides with

$Q$; hence in this case the intersection of $Q$ and the surface (1) is an equilateral hyperbola. From this intersection the hyperbolic functions

$$\sinh u = \frac{e^u - e^{-u}}{2}, \qquad \cosh u = \frac{e^u + e^{-u}}{2},$$
$$\tanh u = \frac{e^u - e^{-u}}{e^u + e^{-u}}, \qquad \text{etc.,} \tag{13}$$

can be given a geometric interpretation in a manner analogous to that often given in calculus books. (For example, Peterson [2].)

First if we let

$$y = \sinh u = \frac{e^u - e^{-u}}{2} \tag{14}$$

and solve for $u$ we get

$$u = \sinh^{-1} y = \log (y + \sqrt{1 + y^2}). \tag{15}$$

The inverse of the other hyperbolic functions (13) can be obtained in a similar manner as

$$\cosh^{-1} y = \log (y \pm \sqrt{y^2 - 1}),$$
$$\tanh^{-1} y = \frac{1}{2} \log \frac{1 + y}{1 - y}, \text{ etc.} \tag{16}$$

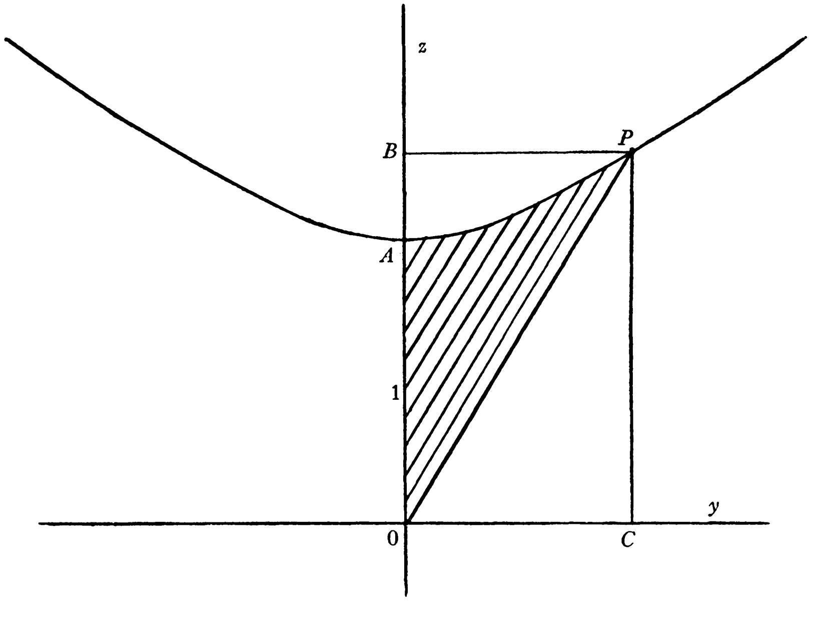


FIG. 5.

Next if we let

$$\tfrac{1}{2}u = \text{Area of } AOP \tag{17}$$

for any point $P$ on the equilateral hyperbola (see Figure 5) then

$$\begin{aligned} \tfrac{1}{2}u &= \int_0^y \sqrt{1+t^2}\,dt - \tfrac{1}{2}zy \\ &= \tfrac{1}{2}\log\left(y+\sqrt{y^2+1}\right) = \tfrac{1}{2}\sinh^{-1} y. \end{aligned} \tag{18}$$

That is, $u = \sinh^{-1} y$ or $\sinh u = y$. From (12) with $a=1$ and from (13) we get that $z = \cosh u$, so the equilateral hyperbola has the parametric representation

$$y = \sinh u \quad \text{and} \quad z = \cosh u. \tag{19}$$

Since the intersection of $Q$ and the surface (1) is a hyperbola for values of $\beta$, $0 \leqq \beta < \pi/4$, then the angle $\Phi$ between the asymptotes varies as $\beta$ varies. Note that the $w$-axis, the intersection of the $Q$ plane and the $z=0$ plane, bisects the angle $\Phi$. Also note that the $w$-axis is the same as the $w$-axis given in Section 2. It readily follows that the asymptotes have the equations

$$z = \pm\, w \tan \Phi/2 \tag{20}$$

in the $zw$-plane. From (12) we get that

$$\tan \Phi/2 = 1/a. \tag{21}$$

(See Figure 6.)

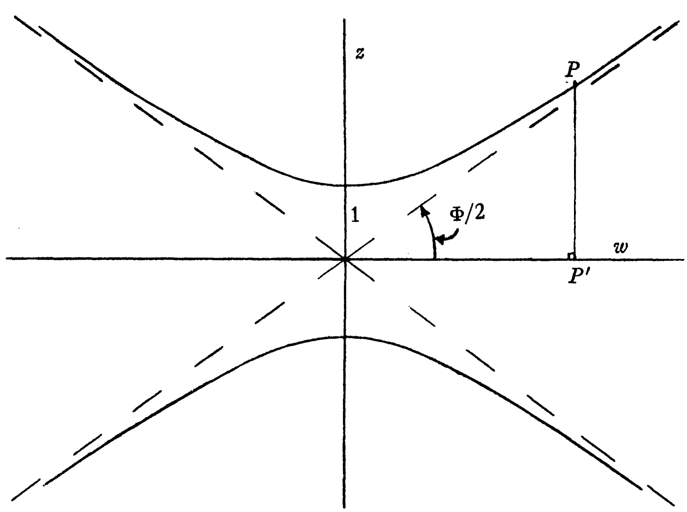


FIG. 6.

Figure 7 shows a portion of a trace of the surface (1) and the trace of $Q$ in the $z=0$ plane as viewed from along the positive $z$-axis with $0 < \beta < \pi/4$. From Figure 6 and Figure 7 we see that if we project a point $P$ from the hyperbola in the $Q$

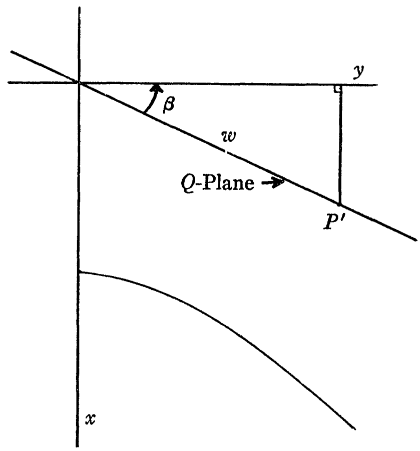


FIG. 7.

plane (not the trace as shown in Figure 7) onto the $w$-axis, then we obtain

$$w^2 = x^2 + y^2. \tag{22}$$

Solving for $a$ in (12), using only the positive square root, and then substituting from (1) and (22) into the result we get

$$a = \frac{x^2 + y^2}{\sqrt{(y^2 - x^2)}}. \tag{23}$$

Then substituting the value of $y$ from (5) into (23) we get that

$$a = \frac{x^2 + x^2 \cot^2 \beta}{\sqrt{(-x^2 + x^2 \cot^2 \beta)}} = \sqrt{(\sec 2\beta)}. \tag{24}$$

From (21) and (24) we get that

$$\tan \Phi/2 = \sqrt{\cos 2\beta}. \tag{25}$$

Thus as the $Q$ plane rotates about the $z$-axis with $0 \leqq \beta < \pi/4$, the angle $\Phi$ between the asymptotes of the hyperbola, which is the intersection of $Q$ and the surface (1), varies from 0 to $\pi/2$; i.e., $\Phi$ is in the interval $0 < \Phi \leqq \pi/2$.

For $\beta$ in the interval $0 < \beta < \pi/4$ we now define, and give a geometric interpretation of, the *quasi-hyperbolic functions*. We define the hyperbolic nis of $u$ or hnis $u$ as

$$\text{hnis } u = \frac{e^{u \tan \Phi/2} - e^{-u \tan \Phi/2}}{2}; \tag{26}$$

the other quasi-hyperbolic functions are defined as

$$\text{hconis } u = \frac{e^{u \tan \Phi/2} + e^{-u \tan \Phi/2}}{2},$$

$$\text{hnat } u = \frac{\text{hnis } u}{\text{hconis } u}, \qquad \text{hconat } u = \frac{1}{\text{hnat } u}, \tag{27}$$

$$\text{hces } u = \frac{1}{\text{hconis } u}, \qquad \text{hcoces } u = \frac{1}{\text{hnis } u}.$$

Note that if $\Phi = \pi/2$ the usual hyperbolic functions are obtained.

First if we let

$$w \tan \Phi/2 = \text{hnis } u = \frac{e^{u \tan \Phi/2} - e^{-u \tan \Phi/2}}{2} \tag{28}$$

and solve for $u$ we get

$$u = \text{hnis}^{-1} (w \tan \Phi/2) = a \log \left( \frac{w + \sqrt{w^2 + a^2}}{a} \right), \tag{29}$$

where from (21) we have $\tan \Phi/2 = 1/a$.

Or if we let

$$z = \text{hconis } u = \frac{e^{u \tan \Phi/2} + e^{-u \tan \Phi/2}}{2} \tag{30}$$

and solve for $u$ we get

$$u = \text{hconis}^{-1} z = a \log (z \pm \sqrt{z^2 - 1}), \tag{31}$$

where $\tan \Phi/2 = 1/a$.

The geometric interpretation of the quasi-hyperbolic functions is similar to that of the hyperbolic functions. If we let

$$\tfrac{1}{2} u = \text{Area of } AOP \tag{32}$$

for any point $P$ on the general hyperbola

$$z^2 - \frac{w^2}{a^2} = 1, \tag{33}$$

(see Figure 8), then

$$\begin{aligned} \tfrac{1}{2} u &= (1/a) \int_0^w \sqrt{(t^2 + a^2)}\, dt - \tfrac{1}{2} z w \\ &= (a/2) \log \left( \frac{w + \sqrt{w^2 + a^2}}{2} \right) = \tfrac{1}{2} \text{ hnis}^{-1} (w \tan \Phi/2), \end{aligned} \tag{34}$$

or

$$w = \frac{\text{hnis } u}{\tan \Phi/2}. \tag{35}$$

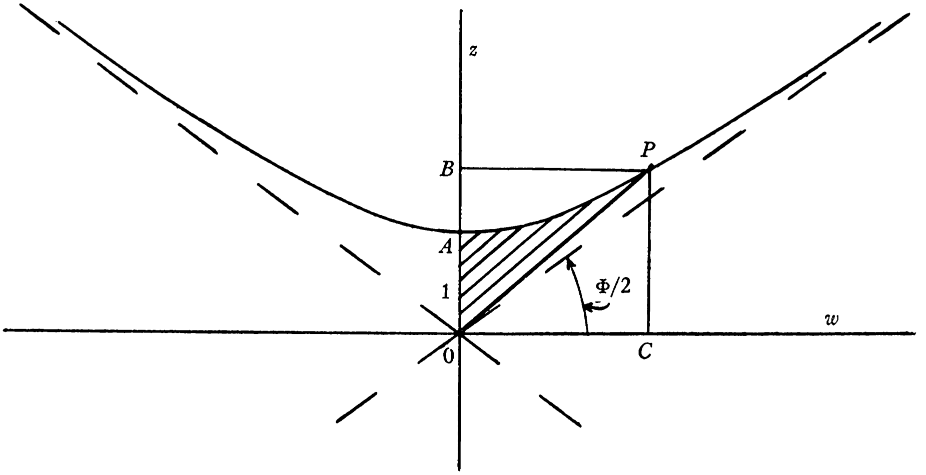


FIG. 8.

From (26), (27), (33), and (35) we get that

$$z = \text{hconis } u. \tag{36}$$

That is, (35) and (36) are the parametric equations of the hyperbola in the rotating $Q$ plane.

Other formulas and identities can be derived in the same manner as in the case of the regular hyperbolic functions which, of course, are the same for $\Phi=\pi/2$.

**4. Quasi-parabolic trigonometry.** For various values of $\beta$ the circular, quasi-elliptic, quasi-hyperbolic, and hyperbolic functions were defined in terms of the intersection of $Q$ and the surface (1). We now consider the limiting case for both the quasi-elliptic and quasi-hyperbolic functions; i.e., we consider the case when the rotating plane makes an angle of $\beta=\pi/4$ with the $y$-axis. In Section 1 we mentioned that the intersection of $Q$ and the surface (1) when $\beta=\pi/4$ is a degenerate parabola, since the hyperboloid of one sheet is a ruled surface. From (10) and (25) we see that, when $\beta=\pi/4, \lambda=0$ and $\Phi=0$. Hence *quasi-parabolic trigonometry* can be interpreted as the limiting case between quasi-elliptic and quasi-hyperbolic trigonometry.

**5. Conclusion.** In (10) we have that $\tan \lambda/2=\sqrt{(-\cos 2\beta)}$ and in (25) we have that $\tan \Phi/2=\sqrt{(\cos 2\beta)}$. From these we get that

$$\tan^2 \lambda/2 = -\tan^2 \Phi/2, \tag{37}$$

or that the relationship between $\lambda$ and $\Phi$ is

$$\lambda = i\Phi. \tag{38}$$

Therefore we again have the tenuous connection found in [3] between the circular and hyperbolic functions.

This article was prepared in an NSF Research Participation for High School Teachers Program, Colorado State University, by Mr. Hapke under the direction of Professor Stein. Mr. Hapke was formerly at Safford High School, Safford, Arizona.

**References**

1. Paula Carter and F. M. Stein, The definition of functions from certain nonlinear differential equations, this MAGAGINE, 38 (1965) 96–101.

2. T. S. Peterson, Analytic Geometry and Calculus, Harper, New York, 1955.

3. Allen Strand and F. M. Stein, Quasi-trigonometry, Amer. Math. Monthly, 69 (1962) 143–147.

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# EXTENSIONS ON A THEME CONCERNING CONDITIONALLY CONVERGENT SERIES

B. J. CERIMELE, Xavier University

**1. Introduction.** The alternating harmonic series has served traditionally in discussions concerning series in the following ways: as an example of a conditionally convergent series, i.e., a series whose convergence depends upon the order as well as the magnitude of its terms, and in illustrating Riemann's Theorem that the sum and the convergence property of a conditionally convergent series can be affected by rearrangement of its terms. Two questions arise in connection with these considerations: (i) what sign distributions on the harmonic series, other than an alternating one, yield convergence? (ii) what is the effect of various rearrangements upon the sum or the convergence property of the alternating harmonic series? These two questions are answered in part for a collection of series whose terms in sequence form the reciprocals of an arithmetic progression, and whose convergence behavior is typified by the alternating harmonic series. Concerning question (i) series having a periodic pattern of sign occurrence are investigated. For such patterns a necessary condition for convergence stipulates that the signs be apportioned evenly in the cyclic sign block. Closed expressions are derived for the series sum in the special cases when the signs alternate in consecutive strings. Relative to question (ii) rearrangements are considered in which the ordering of the terms in the positive and in the negative subseries is maintained. Necessary and sufficient conditions for convergence of such rearranged series are developed, as well as formulas which predict their sum.

**2. Preliminary notions.** From a positive arithmetic progression with a common difference $k$ one can construct the infinite series of reciprocals

$$\sum_{i=0}^{\infty} 1/(j+ik) \tag{1}$$

where $j$ and $k$ denote arbitrary fixed positive integers. For $j=k=1$ this series reduces to the harmonic series. In general the collection of series represented in (1) will be called the $\omega$-series. The following lemma provides certain limit

This article was prepared in an NSF Research Participation for High School Teachers Program, Colorado State University, by Mr. Hapke under the direction of Professor Stein. Mr. Hapke was formerly at Safford High School, Safford, Arizona.

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where $j$ and $k$ denote arbitrary fixed positive integers. For $j=k=1$ this series reduces to the harmonic series. In general the collection of series represented in (1) will be called the $\omega$-series. The following lemma provides certain limit relations involving sequences of portions that are derived from the $\omega$-series, and will prove of use later.

LEMMA 1. *The sequence*

$$\sum_{i=0}^{n-1} 1/(j+ik) - \int_0^n dx/(j+kx) \tag{2}$$

*is positive, monotonically increasing, and bounded above by* $1/j$.

*Proof.* Let $f(x)=1/(j+kx)$ on $[0, n]$; then $f$ is positive and monotonically decreasing. For the partition of the interval $[0, n]$ into $n$ equal subintervals one obtains the inequality chain

$$0 < (1/2)[f(0) - f(n)] < \sum_{i=0}^{n-1} f(i) - \int_0^n f(x)dx < f(0) - f(n).$$

Since $f(n)$ is a null sequence, the above chain shows that the sequence in (2) has the properties predicted in the theorem.

COROLLARY. *Let*

$$E_{jk} = \lim_{n\to\infty}\left[\sum_{i=0}^{n} 1/(j+ik) - (1/k)\ln((j+kn)/j)\right]; \tag{3}$$

*then* $1/2j < E_{jk} < 1/j$.

Let $E_k$ represent $E_{1k}$; then, for $k=1$, $E_1$ becomes the Euler-Mascheroni constant [1, p. 34]. More generally, if $j$ divides $k$, then $E_{jk}=(1/j)E_{k/j}$. The numbers $E_k$ will be called the *extended Euler constants*. A list of approximate values for the first ten extended Euler constants is given in Table 1.

The relationship in (3) of the corollary can be expressed in the form:

$$\sum_{i=0}^{n} 1/(j+ik) \sim (1/k)\ln((j+kn)/j) + E_{jk} \tag{4}$$

where the tilde symbolizes the usual equivalence of sequences.

THEOREM 1. *Let* $(P_n)$, $n\in I^+$, *and* $(Q_n)$, $n\in I^+$, *be two increasing sequences of positive integers for which*

$$P_n > Q_n \quad \textit{and} \quad \lim_{n\to\infty} P_n/Q_n = r;$$

*then*

$$\lim_{n\to\infty} \sum_{i=Q_n+1}^{P_n} 1/(j+2ik) = (1/2k)\ln r. \tag{5}$$

*Proof.* On the basis of (4) one obtains

$$\sum_{i=Q_n+1}^{P_n} 1/(j+2ik) = \left(\sum_{i=0}^{P_n} - \sum_{i=0}^{Q_n}\right) 1/(j+2ik)$$

$$\begin{aligned}&\sim (1/2k)\ln[(j+2P_nk)/j]+E_{j(2k)}\\ &\quad -(1/2k)\ln[(j+2Q_nk)/j]-E_{j(2k)}\\ &=(1/2k)\ln[(j+2P_nk)/(j+2Q_nk)]\\ &\quad \to (1/2k)\ln r \quad \text{as} \quad n\to\infty.\end{aligned}$$

COROLLARY. *For $p$ and $q$ fixed positive integers where $p>q$*

$$\lim_{n\to\infty}\sum_{i=qn+1}^{pn} 1/(j+2ik) = (1/2k)\ln p/q. \tag{6}$$

**3. Main theorems.** Consider the $\omega$-series having alternate positive and negative signs; such a series will be called the alternating $\omega$-series and will be denoted by

$$\omega(j,k) = \sum_{i=0}^{\infty}(-1)^i/(j+ik). \tag{7}$$

THEOREM 2. *The alternating $\omega$-series $\omega(j,k)$ is conditionally convergent for all $j$ and $k$.*

*Proof.* That the series converges is a direct consequence of the alternating series test. That the series does not converge absolutely can be shown by considering the sequence of portions

$$s(2n,n) = \sum_{i=n+1}^{2n} 1/(j+ik).$$

From (6) one obtains that

$$\lim_{n\to\infty} s(2n,n) = (1/k)\ln 2,$$

and exhibition of a nonnull sequence of portions establishes nonconvergence.

THEOREM 3.

$$\begin{aligned}\omega(j,k) &= \int_0^1 x^{j-1}dx/(1+x^k)\\ &= (-1)^{j-1}(r/k)\ln(1+x)\\ &\quad -(2/k)\sum_{i=0}^{q-1}\left[P_i(x)\cos((2i+1)j\pi/k)\right.\\ &\quad \left.-Q_i(x)\sin((2i+1)j\pi/k)\right]\Big|_0^1\end{aligned}$$

*where*

$$\begin{aligned}k &= 2q+r,\ j=1,2,\cdots,k,\ r=0 \text{ or } 1,\\ P_i(x) &= (1/2)\ln\left[x^2-2x\cos((2i+1)\pi/k)+1\right]\\ Q_i(x) &= \arctan\left[(x-\cos((2i+1)\pi/k))/\sin((2i+1)\pi/k)\right].\end{aligned}$$

*Proof.* The rational function

$$g(x) = x^{j-1}/(1 + x^k)$$

has the power expansion

$$g^*(x) = \sum_{i=0}^{\infty} (-1)^i x^{ik+j-1} \tag{8}$$

which is valid in the interval $|x| < 1$. Since $g$ and $g^*$ agree on the unit interval except at $x = 1$, we have

$$\int_0^1 g(x)dx = \int_0^1 g^*(x)dx.$$

Now

$$\int_0^1 g^*(x)dx = \lim_{\epsilon\to 1-} \int_0^{\epsilon} g^*(x)dx. \tag{9}$$

Further, over a compact interval $[0, \delta]$, $0 < \delta < 1$, the series in (8) is majorized by the convergent geometric series of constants $\sum_{i=0}^{\infty} \delta^{ik+j-1}$; hence, by the Weierstrass $M$-test this series is uniformly convergent on $[0, \delta]$. Consequently, the integral in (9) can be evaluated as follows:

$$\lim_{\epsilon\to 1-} \sum_{i=0}^{\infty} (-1)^i \int_0^{\epsilon} x^{ik+j-1}dx = \lim_{\epsilon\to 1-} \sum_{i=0}^{\infty} (-1)^i(\epsilon^{ik+j}/(ik + j))$$

$$= \sum_{i=0}^{\infty} (-1)^i/(j + ik) = \omega(j, k)$$

where the last step is justified by Abel's Theorem on the continuity of power series [**4**, p. 177]. The indefinite integral in the theorem is recorded in Gröbner and Hofreiter [**2**, p. 20].

The following properties of $\omega$ are easily established.

THEOREM 4.
(a) $\omega(k, k) = (1/k)\omega(1, 1) = (1/k) \ln 2$,
(b) $\omega(j, k) = (1/d)\omega(j/d, k/d)$ *where* $d = \text{g.c.d.}\{j, k\}$,
(c) $\omega(j, 1) = (-1)^{j+1}\omega(1, 1) - \sum_{i=1}^{j-1} (-1)^i/(j-i)$,
(d) *for* $j > k > 1$ *and* g.c.d. $\{j, k\} = 1$

$$\omega(j, k) = (-1)^{[j/k]}\omega(j - [j/k]k, k) - \sum_{i=1}^{[j/k]} (-1)^i/(j - ik),$$

*where* $[j/k]$ *denotes the bracket function.*

Some preliminary values of $\omega$ are compiled in Table 3.

Relative to question (i) concerning patterns of sign distribution which yield convergence, the following theorem due to Cesàro imposes a necessary condition on the relative frequency of plus and minus signs for the convergence of an $\omega$-series.

CESÀRO'S THEOREM. *Let $p_n$ and $q_n$ denote the number of positive and the number of negative terms respectively in the first $n$ terms of a series. If the series is conditionally convergent and its sequence of terms in absolute value is monotonically decreasing, then*

$$\lim_{n\to\infty} p_n/q_n = 1$$

*where the limit is known to exist when the terms are of an order of magnitude not less than those of the harmonic series* [3, p. 17].

In a periodic pattern of sign distribution Cesàro's Theorem leads to the requirement that there be in each cyclic block of terms a balance of plus and minus signs. A basic periodic pattern of sign distribution is one in which $p$ successive plus signs are followed by $p$ successive negative signs. The $\omega$-series with such a sign distribution will be called the *$p$-fold alternating $\omega$-series* and will be denoted by

$$\omega_p(j, k) = \sum_{i=0}^{\infty} (-1)^{[i/p]}/(j + ik). \tag{10}$$

THEOREM 5. *The $p$-fold alternating $\omega$-series are conditionally convergent and have sums given by the formula*

$$\omega_p(j, k) = \sum_{i=0}^{p-1} \omega(j + ik, pk). \tag{11}$$

*Proof.* Application of the Dirichlet test yields the convergence of the series. To generate the sum formulas consider the $p$ alternating subseries:

$$\begin{aligned}
\omega(j, pk) &= 1/j - 1/(j + pk) + 1/(j + 2pk) - \cdots \\
\omega(j + k, pk) &= 1/(j + k) - 1/(j + (p + 1)k) \\
&\quad + 1/(j + (2p + 1)k) - \cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\omega(j + (p - 1)k, pk) &= 1/(j + (p - 1)k) - 1/(j + (2p - 1)k) \\
&\quad + 1/(j + (3p - 1)k) - \cdots
\end{aligned}$$

The sum series

$$\begin{aligned}
&[1/j + 1/(j + k) + \cdots + 1/(j + (p - 1)k)] \\
&\quad - [1/(j + pk) + \cdots + 1/(j + (2p - 1)k)] \\
&\quad + [\quad] - [\quad] \cdots
\end{aligned}$$

converges to the sum of the sums of the component subseries as expressed in (11). Furthermore, the brackets can be removed because the resulting series, viz., the $p$-fold alternating $\omega$-series, is convergent.

In Table 3 the bottom row of entries represents the sums of the $p$-fold alternating $\omega$-series $\omega_p(1, 1)$ for values of $p$ from 1 to 10. Closed expressions for a few of the alternating and $p$-fold alternating $\omega$-series are recorded in Table 2.

With regard to question (ii) concerning Riemann's Theorem only those rearrangements which preserve the ordering of the positive and of the negative subseries will be treated; such rearrangements will be called *subordered rearrangements*.

Let $s(n)$ denote the $(n+1)$th partial sum of an alternating $\omega$-series and $t(n)$ the $(n+1)$th partial sum of a subordered rearrangement of the series. Further, let $(p_i), i\in I^+$, and $(q_i), i\in I^+$, denote two sequences of positive integers for which

$$P_n = \sum_{i=1}^{n} p_i \quad \text{and} \quad Q_n = \sum_{i=1}^{n} q_i,$$

and for which the following limits exist:

$$a = \lim_{n\to\infty} P_{n+1}/Q_n, \qquad b = \lim_{n\to\infty} P_n/Q_n. \tag{12}$$

Consider a rearrangement of the alternating $\omega$-series in which $p_i$ successive positive terms are followed by $q_i$ successive negative terms in the $i$-th set of selections. Then the upper sum $u$ and the lower sum $l$ of the rearranged series are given by

$$u = \lim_{n\to\infty} t(P_{n+1} + Q_n), \qquad l = \lim_{n\to\infty} t(P_n + Q_n)$$

provided that the limits exist. For $P_{n+1} > Q_n (Q_n > P_{n+1})$

$$t(P_{n+1} + Q_n) = s(2Q_n) + \sum_{i=Q_n+1}^{P_{n+1}} 1/(j + 2ik)$$

$$\left(= s(2Q_n) - \sum_{i=P_{n+1}+1}^{Q_n} 1/(j + 2ik)\right)$$

and consequently, by virtue of (5),

$$\lim_{n\to\infty} t(P_{n+1} + Q_n) = \omega(j, k) + (1/2k) \ln a.$$

Furthermore, for $P_n > Q_n (Q_n > P_n)$

$$t(P_n + Q_n) = s(2Q_n) + \sum_{i=Q_n+1}^{P_n} 1/(j + 2ik)$$

$$\left(= s(2Q_n) - \sum_{i=P_n+1}^{Q_n} 1/(j + 2ik)\right)$$

and consequently, by virtue of (5),

$$\lim_{n\to\infty} t(P_n + Q_n) = \omega(j, k) + (1/2k) \ln b.$$

Hence, the upper and lower sums become

$$(13) \qquad u = \omega(j, k) + (1/2k) \ln a, \qquad l = \omega(j, k) + (1/2k) \ln b$$

when $a$ and $b$ in (12) exist.

From the above discussion Theorems 6 and 7 follow easily.

THEOREM 6. *A subordered rearrangement of the alternating $\omega$-series $\omega(j, k)$ is convergent if and only if the limits $a$ and $b$ exist and are equal. The sum of the rearranged series is given by*

$$\omega(j, k) + (1/2k) \ln b.$$

THEOREM 7. *A subordered rearrangement of the alternating $\omega$-series remains convergent if and only if*

$$\lim_{n\to\infty} p_n/P_{n-1} = \lim_{n\to\infty} q_n/Q_{n-1} = 0.$$

*Proof.* From (13) one obtains

$$\begin{aligned} u - l &= (1/2k) \ln(a/b) \\ &= (1/2k) \ln(\lim_{n\to\infty} P_{n+1}/P_n) \\ &= (1/2k) \ln(1 + \lim_{n\to\infty} p_{n+1}/P_n) \end{aligned}$$

and also

$$\begin{aligned} &= (1/2k) \ln(\lim_{n\to\infty} Q_{n+1}/Q_n) \\ &= (1/2k) \ln(1 + \lim_{n\to\infty} q_{n+1}/Q_n) \end{aligned}$$

which vanishes when and only when the condition in the theorem holds.

COROLLARY. *A subordered rearrangement of the alternating $\omega$-series leaves the sum unaffected if and only if $a = b = 1$.*

Special cases of interest are the following:

a) If for every $i$ $p_i = p$ and $q_i = q$, then the sum of the rearranged series is given by

$$\omega(j, k) + (1/2k) \ln p/q.$$

b) If for every $i$ $p_i = q_i$ and $a$ exists, then

$$u = \omega(j, k) + (1/2k) \ln (1 + c), \qquad l = \omega(j, k)$$

where

$$c = \lim_{n\to\infty} p_{n+1}/P_n.$$

c) If for every $i$ $p_{i+1} = q_i$ and $b$ exists, then

$$u = \omega(j, k), \qquad l = \omega(j, k) + (1/2k) \ln(1 - d)$$

where

$$d = \lim_{n\to\infty} q_n/Q_n.$$

## 4. Tabular data.

TABLE 1. SOME EXTENDED EULER CONSTANTS $E_k$

| | | | |
|---|---|---|---|
| $E_1$ | .577216 | $E_6$ | .756624 |
| $E_2$ | .634869 | $E_7$ | .773921 |
| $E_3$ | .677599 | $E_8$ | .788553 |
| $E_4$ | .710134 | $E_9$ | .801122 |
| $E_5$ | .735795 | $E_{10}$ | .812054 |

$E_1$ is the Euler-Mascheroni constant.

$E_2$ through $E_{10}$ were estimated on an IBM 1620 digital computer by calculating 800 terms of the defining sequence in (3) using 15 digit mantissas. The maximum variation among the last few terms computed was of the order $10^{-5}$.

TABLE 2. CLOSED EXPRESSIONS FOR THE SUMS OF THE ALTERNATING AND $p$-FOLD ALTERNATING $\omega$-SERIES

| | $k$ | | | |
|---|---|---|---|---|
| $\omega(j, k)$ | 1 | 2 | 3 | 4 |
| $j$ 1 | ln 2 | $\pi/4$ | $(1/9)[\pi\sqrt{3}+3\ln 2]$ | $(\sqrt{2}/8)[\pi+2\ln(1+\sqrt{2})]$ |
| $j$ 2 | | $(1/2)\ln 2$ | $(1/9)[\pi\sqrt{3}-3\ln 2]$ | $\pi/8$ |
| $j$ 3 | | | $(1/3)\ln 2$ | $(\sqrt{2}/8)[\pi-2\ln(1+\sqrt{2})]$ |
| $j$ 4 | | | | $(1/4)\ln 2$ |
| $\omega_p(1, 1)$ | ln 2 | $(1/4)[\pi+2\ln 2]$ | $(1/9)[2\pi\sqrt{3}+3\ln 2]$ | $(1/8)[\pi(1+2\sqrt{2})+2\ln 2]$ |

TABLE 3. SUMS OF THE ALTERNATING AND $p$-FOLD ALTERNATING $\omega$-SERIES

| $\omega(j, k)$ | $k$ = 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $j$ 1 | .693147 | .785398 | .835649 | .866973 | .888314 | .903772 | .915480 | .924652 | .932030 | .938094 |
| $j$ 2 | | .346574 | .373551 | .392699 | .406902 | .417824 | .426474 | .433486 | .439285 | .444157 |
| $j$ 3 | | | .231049 | .243748 | .253752 | .261799 | .268398 | .273898 | .278550 | .282532 |
| $j$ 4 | | | | .173287 | .180646 | .186775 | .191943 | .196350 | .200147 | .203451 |
| $j$ 5 | | | | | .138629 | .143426 | .147562 | .151156 | .154304 | .157080 |
| $j$ 6 | | | | | | .115525 | .118897 | .121874 | .124517 | .126876 |
| $j$ 7 | | | | | | | .099021 | .101520 | .103766 | .105790 |
| $j$ 8 | | | | | | | | .086643 | .088570 | .090323 |
| $j$ 9 | | | | | | | | | .077016 | .078546 |
| $j$ 10 | | | | | | | | | | .069315 |
| $\omega_p(1, 1)$ | .693147 | 1.131972 | 1.440249 | 1.676707 | 1.868242 | 2.029121 | 2.167773 | 2.289580 | 2.398184 | 2.496164 |
| $p$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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# INFINITE CLASSES OF HARMONIC INTEGERS

J. W. BERGQUIST and LORRAINE L. FOSTER, IBM-California Institute of Technology

1. The well known theorem of Lagrange [1] states that every positive integer is the sum of the squares of four integers. Moreover, it is known from projective geometry that three points $x$, $y$, and $u$ on a line in the projective plane uniquely determine a fourth point $v$ known as the fourth harmonic point. It is equivalent to the geometrical construction that these four points (a harmonic set) have, when taken in their proper order, a cross ratio $R(x, y; u, v) = -1$ Cross ratio is defined [2] by:

$$R(x, y; u, v) = \frac{v-x}{v-y}\cdot\frac{u-y}{u-x}$$

Based upon these two ideas, the concept of the harmonic integer can be established. It is the main result of this paper that there exist explicitly computable, infinite classes of harmonic integers.

DEFINITION 1.1. *Let $N=a^2+b^2+c^2+d^2$ where $R(a^2, b^2; c^2, d^2) = -1$. Then $N$ is a harmonic integer and $a^2+b^2+c^2+d^2$ is a harmonic representation of $N$.*

Since $R((ka)^2, (kb)^2; (kc)^2, (kd)^2) = R(a^2, b^2; c^2, d^2)$ it is convenient to make the following definition:

DEFINITION 1.2. *The representation $a^2+b^2+c^2+d^2$ of $N$ is a reduced harmonic representation of $N$ if and only if it is a harmonic representation of $N$ such that* g.c.d. $\{a, b, c, d\} = 1$.

We observe that if $a^2+b^2+c^2+d^2$ is a harmonic representation of $N$, then, by suitable permutation of $a, b, c, d$ we can find integers $a_1, b_1, c_1, d_1$, such that $a_1^2+b_1^2+c_1^2+d_1^2$ is a harmonic representation of $N$ and $0 \leqq a_1 < c_1 < b_1 < d_1$. Thus we are led to make the following definition:

DEFINITION 1.3. *An integer $N$ with harmonic representation $a^2+b^2+c^2+d^2$ is in standard form if and only if $0 \leqq a < c < b < d$.*

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Based upon these two ideas, the concept of the harmonic integer can be established. It is the main result of this paper that there exist explicitly computable, infinite classes of harmonic integers.

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Since $R((ka)^2, (kb)^2; (kc)^2, (kd)^2) = R(a^2, b^2; c^2, d^2)$ it is convenient to make the following definition:

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We observe that if $a^2+b^2+c^2+d^2$ is a harmonic representation of $N$, then, by suitable permutation of $a$, $b$, $c$, $d$ we can find integers $a_1$, $b_1$, $c_1$, $d_1$, such that $a_1^2+b_1^2+c_1^2+d_1^2$ is a harmonic representation of $N$ and $0 \leq a_1 < c_1 < b_1 < d_1$. Thus we are led to make the following definition:

DEFINITION 1.3. *An integer* $N$ *with harmonic representation* $a^2+b^2+c^2+d^2$ *is in standard form if and only if* $0 \leq a < c < b < d$.

Some examples of harmonic integers in reduced, standard form are $1299 = 0^2+7^2+5^2+35^2$, $1416243 = 0^2+41^2+29^2+1189^2$, and $147 = 1^2+4^2+3^2+11^2$. An infinite set of integers with reduced harmonic representations which contains 1299 and 1416243 is given by the following theorem:

THEOREM 1.1. *Define integers $a_k$, $b_k$, $c_k$, $d_k$ by the relations $a_k=0$, $b_k+\sqrt{2}c_k = (1+\sqrt{2})(3+2\sqrt{2})^k$, and $d_k=b_kc_k$. Then $N_k=a_k^2+b_k^2+c_k^2+d_k^2$ is a reduced harmonic representation of $N_k$ in standard form.*

*Proof.* We observe that $b_k^2-2c_k^2=-1$, $k=1, 2, \cdots$, $5=c_1<c_2<\cdots$, and $7=b_1<b_2<\cdots$. Hence

$$R(a_k^2, b_k^2; c_k^2, d_k^2) = -1 \tag{1.1}$$

and g.c.d. $\{b_k, c_k\}=1$, $k=1, 2, \cdots$. Clearly $0=a_k<c_k<b_k<d_k$ so that, from (1.1), $N_k$ is in standard form.

The harmonic integer 147 given above is an element of the infinite set of integers with reduced harmonic representations given by the following theorem.

THEOREM 1.2. *There exist infinitely many integers of the form $3x^2$ having reduced harmonic representations.*

*Proof.* Suppose that there exist positive integers $A$, $B$ and $x$ such that

$$3A^2 + B^2 = x^2, \qquad B \neq A, x. \tag{1.2}$$

Then

$$3x^2 = A^2 + B^2 + (x - A)^2 + (x + A)^2 \tag{1.3}$$

and it can easily be verified that $R(A^2, B^2; (x-A)^2, (x+A)^2) = -1$. We recall [3] that if $x$ is a prime $p$ of the form $6n+1$, then equation (1.2) has a solution $(A, B) = (A_p, B_p)$ with $B_p \neq A_p, p$. Then (1.3) gives us a reduced harmonic representation of $3p^2$.

Consider the integer 8112 which has the following harmonic representations, in standard form:

$$2^2 + 30^2 + 22^2 + 82^2, \qquad 14^2 + 46^2 + 38^2 + 66^2,$$
$$12^2 + 52^2 + 44^2 + 68^2, \qquad 16^2 + 44^2 + 36^2 + 68^2$$

The second, third and fourth representations can be obtained from the first by successive applications of the following theorem with $(a, b, c, d) = (2, 30, 22, 82)$, $(-2, 30, 22, 82)$, $(-2, 30, -22, 82)$, and suitable permutations. Verification of the theorem is left to the reader.

THEOREM 1.3. *Let $a^2+b^2+c^2+d^2$ be a harmonic representation of an even integer $N$. Then $a_1^2+b_1^2+c_1^2+d_1^2$ is also a harmonic representation of $N$ where $a_1 = (b + c - a - d)/2$, $b_1 = (b + d - a - c)/2$, $c_1 = (d + c - a - b)/2$, and $d_1 = (a+b+c+d)/2$.*

We have an immediate corollary to Theorem 1.3:

COROLLARY 1.4. *Let $N$ be an even integer with harmonic representation $a^2+b^2$*

$+c^2+d^2$ *in standard form. Then if* $d \neq a+b+c$, *N has at least one other distinct harmonic representation in standard form.*

*Proof.* We take $a_1$, $b_1$, $c_1$, $d_1$ as in the theorem and observe that $d_1 \notin \{a, b, c, d\}$. An IBM 7094 program was written to calculate all harmonic integers such that $0 \leqq a < c < b \leqq 100$. A total of 169 standard forms was calculated (some representing the same number). All but 14 of these are multiples of forms given with suitable permutation by Theorems 1.1, 1.2, and 1.4. These 14 integers $N$ are tabulated below.

TABLE I

| $N$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| 3247 | 19 | 29 | 26 | 37 |
| 3847 | 1 | 34 | 29 | 43 |
| 9469 | 10 | 55 | 50 | 62 |
| 20593 | 12 | 44 | 33 | 132 |
| 28927 | 25 | 47 | 38 | 157 |
| 32419 | 16 | 79 | 61 | 149 |
| 36253 | 46 | 73 | 62 | 158 |
| 37951 | 42 | 93 | 77 | 147 |
| 45127 | 11 | 29 | 22 | 209 |
| 65523 | 17 | 52 | 39 | 247 |
| 73999 | 7 | 95 | 70 | 245 |
| 84031 | 13 | 78 | 57 | 273 |
| 1308463 | 5 | 86 | 61 | 1139 |
| 1542319 | 21 | 66 | 49 | 1239 |

It is noted that all of these numbers except 65523 are of the form $6n+1$, and that 3847 is the smallest prime harmonic integer.

2. It seems of interest to generalize the notion of a harmonic integer by defining

DEFINITION 2.1. $H_n = \{N = a^2+b^2+c^2+d^2 \mid R(a^2, b^2; c^2, d^2) = -n\}$, $n \in I$. (Clearly $N$ is a harmonic integer in the sense of Section 1 if and only if $N \in H_1$.)

We also define two subsets of $H_n$ as follows:

DEFINITION 2.2.

$$H_n' = \{N = a^2 + b^2 + c^2 + d^2 \mid N \in H_n, abcd \neq 0\}$$
$$H_n'' = \{N = a^2 + b^2 + c^2 + d^2 \mid N \in H_n, a = 0\}.$$

From Theorem 1.1 and the proof of Theorem 1.2 we can see that $H_1'$ and $H_1''$ are nonempty (and, in fact, contain infinitely many integers with reduced harmonic representations). In Theorems 2.1 and 2.2 we demonstrate that for any $n$, $H_n'$ and $H_n''$ are nonempty. Theorem 2.3 is a stronger theorem for the case $n=2$.

THEOREM 2.1. $H_n'' \neq \phi$, *for all* $n \in I$.

*Proof.* We have observed that the theorem is true in the case $n=1$ by Theorem 1.1. Also, we note that $10243=39^2+21^2+91^2\in H_3''$ and $9=1^2+2^2+2^2 \in H_{-1}''$. Hence we can assume that $n\neq \pm 1$, 3. Let $b=(3n-1)(n-3)$, $c=(n-3) \cdot(n+1)$, and $d=(3n-1)(n+1)$. Then one can easily verify that for $n\neq \pm 1$, 3, the integer $b^2+c^2+d^2\in H_n''$.

THEOREM 2.2. *$H_n' \neq \phi$ for all $n\in I$.*

*Proof.* We observe the relation

$$(2.1) \qquad (n^2 + n + 1)x^2 = A^2 + B^2 + (x - A)^2 + (nx + A)^2$$

where

$$(2.2) \qquad B^2 = nx^2 - 2(n - 1)xA - 3A^2.$$

We also observe that, for any $n$, if $A$, $B$ and $x$ are nonzero integers satisfying (2.1), (2.2), and the inequalities

$$(2.3) \qquad A \neq -xn/2, \quad x/2, \quad x(1 - n)/2,$$

then $N=(n^2+n+1)x^2\in H_n$. For, if we choose $a=A$, $b=B$, $c=x-A$, $d=nx+A$, then $R(a^2, b^2; c^2, d^2)=-n$. Hence to prove the theorem it is sufficient to exhibit nonzero integers $x$, $A$ and $B$ such that (2.2), (2.3), and

$$(2.4) \qquad (x - A)(nx + A) \neq 0$$

are satisfied. We observe that we can choose $x=-6$, $A=n-1$, and $B=3(n+1)$. Then, if $n\neq 1$, $-2$, $-5$, conditions (2.2), (2.3), and (2.4) are fulfilled. Finally since $147=1^2+4^2+3^2+11^2\in H_1'$, $147=1^2+11^2+3^2+4^2\in H_{-2}'$, and $84=1^2+7^2+3^2+5^2\in H_{-5}'$, the theorem is true for all $n$.

THEOREM 2.3. *There are infinitely many distinct integers with reduced harmonic representations in $H_2'$ and in $H_2''$.*

*Proof.* Setting $n=2$ in conditions (2.2), (2.3), and (2.4) we have:

$$(2.2') \qquad B^2 = 2x^2 + 2Ax - 3A^2$$

$$(2.3') \qquad A \neq -x, \pm x/2$$

$$(2.4') \qquad A \neq x, -2x.$$

We observe that equation (2.2′) can be written in the form:

$$(2.2'') \qquad 2B^2 = y^2 - 7A^2$$

where $y=2x+A$. Suppose that for some fixed $B>0$ this equation has a solution $y_0$, $A_0$ with $y_0, A_0>0$. Then the integers $y_k$, $A_k$ defined by the relation

$$y_k + \sqrt{7}A_k = (y_0 + \sqrt{7}A_0)(8 + 3\sqrt{7})^k, \quad k = 1, 2, \cdots,$$

will also be solutions [4] of (2.2″). Further, we can define integers $x_k$ by

$$x_k = (y_k - A_k)/2, \quad k = 1, 2, \cdots.$$

One can easily verify that since $A_1<A_2< \cdots$, we can choose an integer $k$ such that for $k \geqq k_0$, (2.2′), (2.3′), and (2.4′) are satisfied with $x = x_k$, $A = A_k$. Hence, from the proof of Theorem 2.2 we know that the integer

$$N_k = A_k^2 + B^2 + (x_k - A_k)^2 + (2x_k + A_k)^2 \tag{2.5}$$

is in $H_2'$ for $k \geqq k_0$. To prove that $H_2'$ contains infinitely many reduced forms it is therefore sufficient to note that $(B, y_0, A_0) = (1, 3, 1)$ is a solution of (2.2″) and the corresponding representations of integers $N_k$ given by equation (2.5) are reduced.

To prove that $H_2''$ contains infinitely many reduced forms we define integers $b_k$, $c_k$, $d_k$ by

$$b_k + \sqrt{3}c_k = (5 + 3\sqrt{3})(2 + \sqrt{3})^{k-1} \quad \text{and} \quad d_k = b_k c_k \quad k = 1, 2, \cdots.$$

Then $N_k = b_k^2 + c_k^2 + d_k^2 \in H_2''$ and $b_k^2 + c_k^2 + d_k^2$ is in reduced form since $b_k^2 - 3c_k^2 = -2$, $b_k \equiv c_k \equiv 1 \pmod 2$.

Another direction in which we can generalize the notion of a harmonic integer is indicated by the following definition:

DEFINITION 2.3. *Suppose* $N = a^3 + b^3 + c^3 + d^3$ and $R(a^3, b^3; c^3, d^3) = -1$. *Then we say* $N$ *is a cube-harmonic integer.*

An examination of over $7 \times 10^5$ triples $(a, b, c)$, performed on the IBM 7094, revealed only cube-harmonic integers of the form given in the following theorem.

THEOREM 2.4. *Let* $x$ *and* $y$ *be nonzero integers such that* $xy = \square$, $x \neq y$. *Then* $x^3 + y^3$ *is cube-harmonic.*

*Proof.* Let $d = x$, $c = y$, $b = +\sqrt{(xy)}$, $a = -b$.

**References**

1. G. H. Hardy and E. M. Wright, The Theory of Numbers, 4th ed., Oxford, New York, 1960.
2. Herbert Busemann and P. J. Kelly, Projective Geometry and Projective Metrics, Academic Press, New York, 1953.
3. T. Nagell, Introduction to Number Theory, Ch. 6, Wiley, New York, 1951.
4. W. J. Le Veque, Topics in Number Theory, vol. I, Addison-Wesley, Reading, 1956.

## PROPERTIES OF THE COMPLETE PENTAGON

WILLIAM H. BUNCH, College Place, Washington

Much has been published about the intriguing properties of the complete quadrilateral, but little has been said about a complete figure formed from five lines. I shall confine my discussion to the case in which we have given five lines, no two of which are parallel and no three of which are concurrent.

Given any five lines, $AB$, $GF$, $HI$, $AG$, and $BF$, (Figure 1), we may select any four as $GF$, $HI$, $AG$, and $BF$, to form a quadrilateral. The ordinary quadrilateral formed by these lines is the figure $FGHI$ which has two pairs of opposite that for $k \geqq k_0$, (2.2′), (2.3′), and (2.4′) are satisfied with $x = x_k$, $A = A_k$. Hence, from the proof of Theorem 2.2 we know that the integer

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$$b_k + \sqrt{3}c_k = (5 + 3\sqrt{3})(2 + \sqrt{3})^{k-1} \quad \text{and} \quad d_k = b_k c_k \quad k = 1, 2, \cdots.$$

Then $N_k = b_k^2 + c_k^2 + d_k^2 \in H_2''$ and $b_k^2 + c_k^2 + d_k^2$ is in reduced form since $b_k^2 - 3c_k^2 = -2$, $b_k \equiv c_k \equiv 1 \pmod 2$.

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*Proof.* Let $d = x$, $c = y$, $b = +\sqrt{(xy)}$, $a = -b$.

**References**


1. G. H. Hardy and E. M. Wright, The Theory of Numbers, 4th ed., Oxford, New York, 1960.
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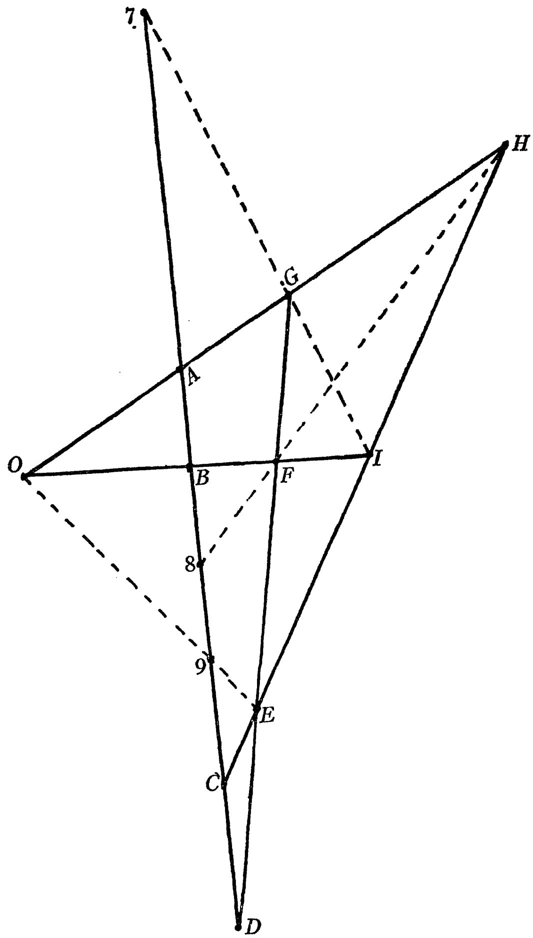


FIG. 1.

vertices: $G$, $I$ and $H$, $F$. The complete quadrilateral formed by these lines, however, extends to $O$ and $E$ which are considered to be opposite vertices. The three diagonals of this quadrilateral will be the lines, $IG$, $HF$, and $OE$. These in turn determine three points, 7, 8, and 9, on the fifth given line, $AB$. Now in this same way, by holding out a different one of the given lines each time, we determine five quadrilaterals each of which will determine three points on a different given line. Since there are five lines there will be fifteen such points which we shall call the $V$ points, (see Figure 1).

Now we wish to show that the fifteen $V$ points, which are represented by numbers, are connected by twenty-five lines, including the five given lines, in such a way that five lines pass through each point and three points lie on each line. We shall call these twenty-five lines, the $v$ lines. Five $v$ lines are the given lines and the other twenty $v$ lines are not immediately defined by the given lines. The fifteen $V$ points are first defined as in Figure 1; then the $v$ lines are defined by the property of these points.

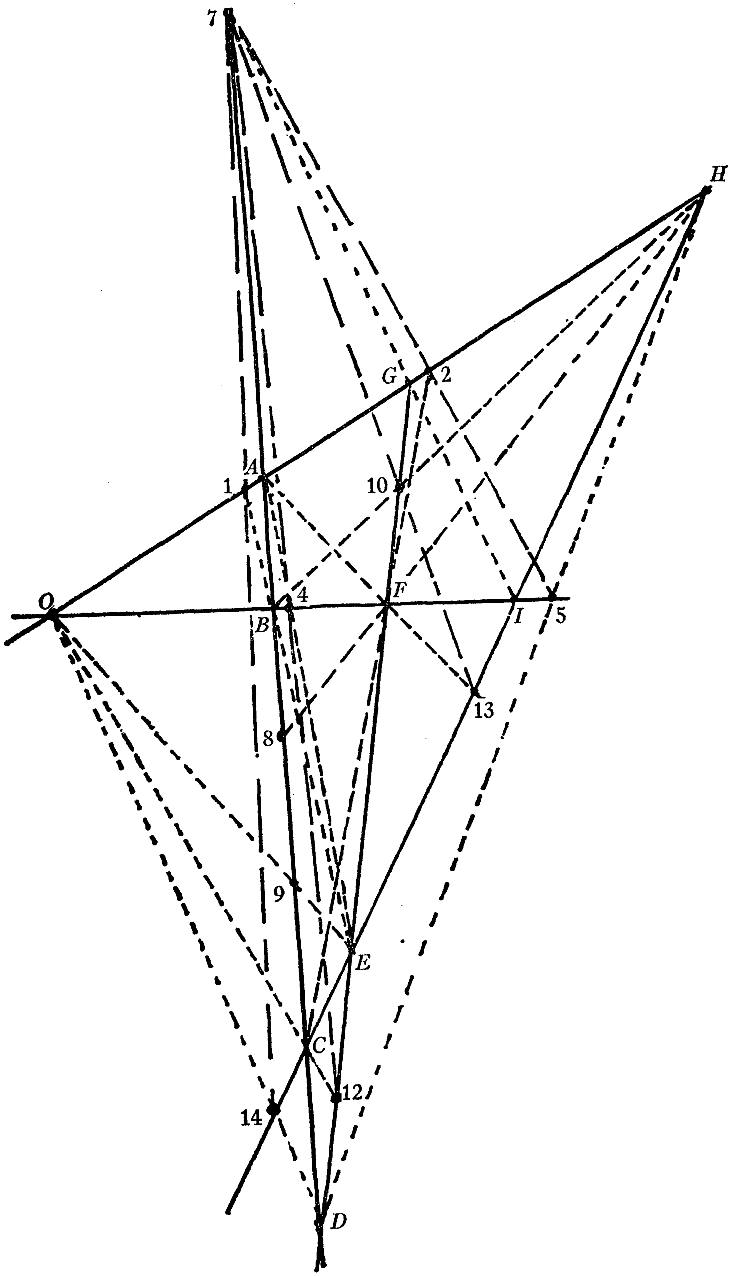


FIG. 2.

We have stated above that diagonal $IG$ of quadrilateral $FGHI$ defines point 7 on line $AB$ (Figure 2). Also diagonal $AF$ of quadrilateral $ABFG$ defines point 13 on line $HI$. Diagonal $BH$ of quadrilateral $ABIH$ defines point 10 on line $FG$.

Now let lines *OH* and *OI* be a degenerate conic with points *H*, *A*, and *G* on *OH* and points *F*, *I*, and *B* on *OI*; then by Pascal's Theorem the points defined by the intersections of lines *AF* and *HI*, *HB* and *FG*, and *IG* and *AB* lie on a line. But these are the points 13, 10, and 7. Hence the line 13, 10, 7 is a *v* line.

Let diagonal *DH* of quadrilateral *ACEG* determine point 5 on line *OI*. Let diagonal *CF* of quadrilateral *BCEF* determine point 2 on line *OH*. Now let lines *EG* and *EH* be the conic with points *C*, *H*, and *I* on *EH* and *G*, *F*, and *D*, the three points on *EG*. Then the points determined by the intersections of the three pairs of lines, *HD* and *FI*, *CF* and *GH*, and *CD* and *GI*, lie on a line. Hence 5, 2, and 7 give us another three *V* points on a *v* line.

We now use diagonal *OC* of quadrilateral *ABIH* to determine point 12 on line *GE* and diagonal *AE* of quadrilateral *ACEG* to determine point 4 on line *OI*. As a conic use lines, *HO* and *HE*, with points *G*, *A*, and *O*, on line *HO* and *C*, *I*, and *E* on line *HE*. Then the points determined by the intersection of lines *GE* and *CO*, *AE* and *IO*, and *GI* and *AC*, lie on a line. But these are the *V* points 12, 4 and 7 making another *v* line through 7.

Again let diagonal *OD* of quadrilateral *ABFG* determine point 14 on line *HE*, and diagonal *BE* of quadrilateral *BCEF* determine point 1 on line *OH*. Let lines *GE* and *OI* be the conic with points *O*, *I*, and *B* on *OI*, and points *D*, *E*, and *G* on *GE*. The points determined by the intersections of the three pairs of lines *IE* and *DO*, *EB* and *OG*, and *IG* and *DB* lie on a line, or line 14, 1, 7 makes another *v* line through 7.

We have now examined eleven of the fifteen *V* points. One of the other four lies on each of four given lines on which point 7 does not lie. These taken two at a time could form six lines. The same type of investigation as used above shows that two of these cross line *AB* at 8 and two at 9 and the other two are not *v* lines. Hence five and only five *v* lines pass through point 7. The similarity in the way the *v* points are determined shows that any point could be used for point 7 and that five *v* lines pass through every *V* point. Now since three points lie on each line we have a complete figure consisting of twenty-five lines and fifteen points with five lines passing through each point and three points lying on each line. In this way the simple pentagon becomes the complex figure shown in Figure 3.

Now let us choose two of the original five lines, *OH* with *V* points 1, 2, and 3 on it and *OI* with *V* points 4, 5, and 6 on it, for a conic. Let the notation 1, 6 represent the line joining points 1 and 6. Then 2, 5 will be the line joining points 2 and 5. The notation 1, 6–2, 5 will represent a point, the intersection of lines 1, 6 and 2, 5. A degenerate conic composed of two lines with three points on each will produce six Pascal Lines and no more.

For the lines we have chosen, the Pascal Lines are:

| *a* | *b* | *c* | *d* | *e* | *f* |
|---|---|---|---|---|---|
| 1, 6–2, 5 | 1, 6–3, 4 | 3, 4–2, 5 | 1, 5–2, 6 | 3, 5–1, 6 | 1, 6–2, 4 |
| 3, 6–2, 4 | 2, 6–3, 5 | 1, 4–2, 6 | 3, 5–2, 4 | 2, 6–3, 4 | 3, 6–2, 5 |
| 1, 4–3, 5 | 2, 4–1, 5 | 1, 5–3, 6 | 1, 4–3, 6 | 2, 5–1, 4 | 1, 5–3, 4 |

Let us pick a point from each of the first three: 1, 6–2, 5 from *a*, 1, 6–3, 4

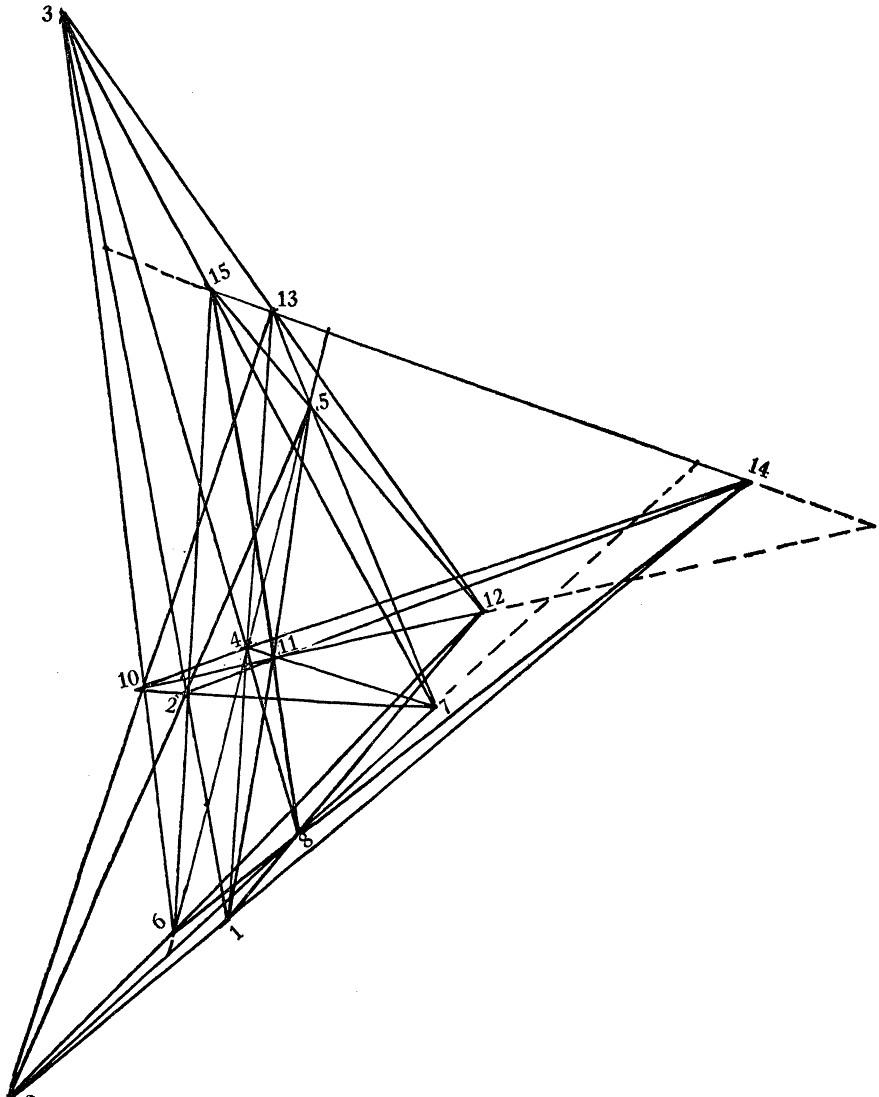


Fig. 3.

from $b$, and 3, 4–2, 5 from $c$. Obviously the line 1, 6 joins two Pascal Points, 3, 4 joins two, and also 2, 5 joins two. We have now a triangle with sides 1, 6; 2, 5; and 3, 4, and whose vertices are a Pascal Point on each Pascal Line. Again let us choose three Pascal Points from the same lines: 3, 6–2, 4 from $a$, 2, 4–1, 5 from $b$, and 1, 5–3, 6 from $c$. Again we have a triangle whose sides are 1, 5; 2, 4; and 3, 6 and whose vertices are different Pascal Points on the same three Pascal Lines. The corresponding sides of these triangles are: 1, 5; 1, 6 and 2, 4, and 2, 5; 3, 6 and 3, 4. These obviously meet in the points 1, 2, and 3 which are on the line $OH$. Therefore, by Desargues' Theorem, Pascal Lines $a$, $b$, and $c$ meet in a point. The same observation would show that lines $d$, $e$, and $f$ also meet in a point. (See Figure 4.)

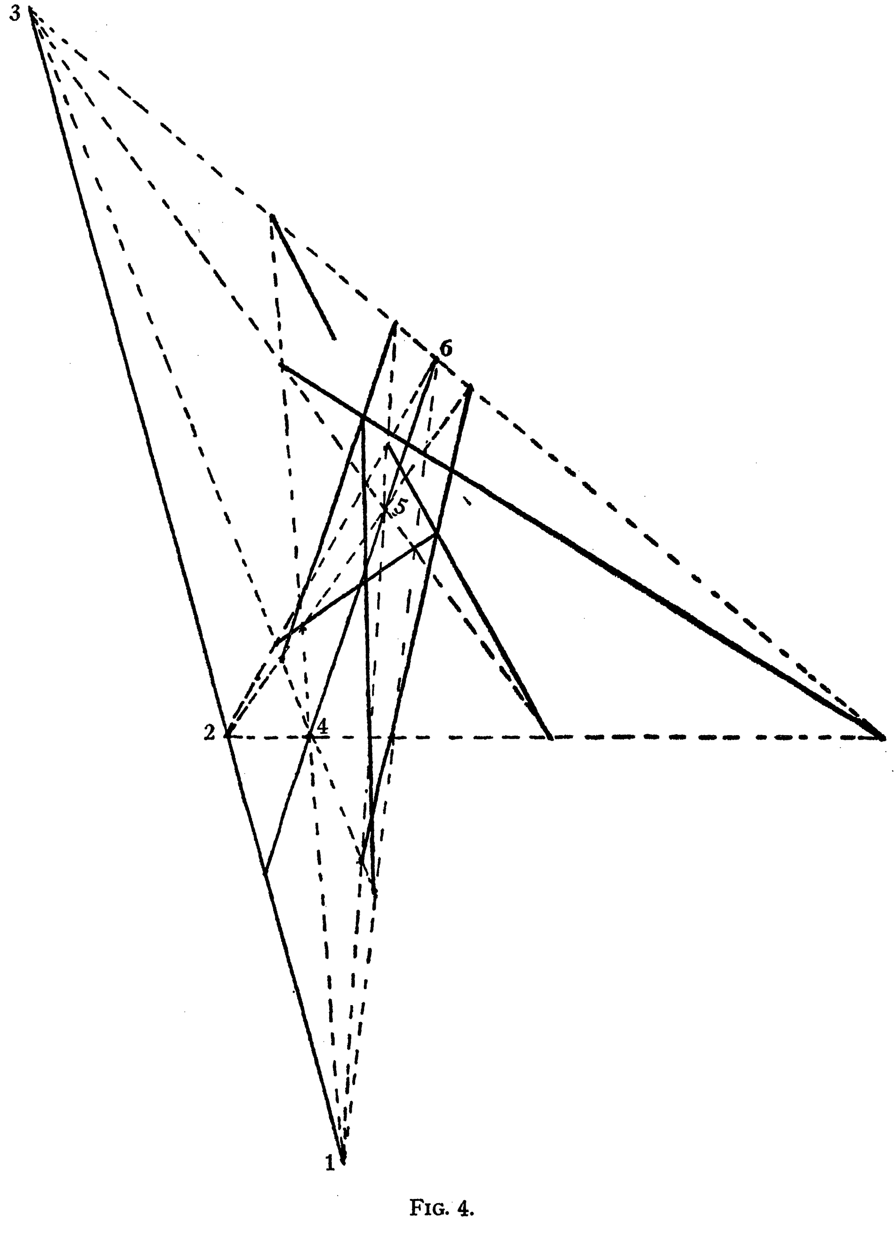


FIG. 4.

We shall call these six Pascal Lines made from a pair of straight lines used as a conic, a Pascal group. Since five lines can be paired into ten pairs of two each we may have ten Pascal groups or a total of sixty lines. Since each group contains two sets of three lines which intersect in a point, the sixty lines will intersect by threes in twenty points. Since any Pascal Point can be used as above as the vertex of a triangle in applying Desargues' Theorem none of the twenty points will be Pascal Points.

The sixty lines divided into the twenty sets of three each are listed below for further use. These are followed by the twenty-five $v$ lines with three $V$ points each.

| | | | | | |
|---|---|---|---|---|---|
| | 1 | | | 2 | |
| 1, 6–2, 5 | 1, 6–3, 4 | 3, 4–2, 5 | 1, 5–2, 6 | 3, 5–1, 6 | 1, 6–2, 4 |
| 3, 6–2, 4 | 2, 6–3, 5 | 1, 4–2, 6 | 3, 5–2, 4 | 2, 6–3, 4 | 3, 6–2, 5 |
| 1, 4–3, 5 | 2, 4–1, 5 | 1, 5–3, 6 | 1, 4–3, 6 | 2, 5–1, 4 | 1, 5–3, 4 |
| | 3 | | | 4 | |
| 1, 9–2, 8 | 1, 7–2, 9 | 3, 7–1, 9 | 1, 8–2, 9 | 3, 8–1, 9 | 1, 9–2, 7 |
| 3, 9–2, 7 | 3, 7–2, 8 | 2, 9–3, 8 | 3, 8–2, 7 | 2, 9–3, 7 | 3, 9–2, 8 |
| 1, 7–3, 8 | 1, 8–3, 9 | 2, 7–1, 8 | 1, 7–3, 9 | 2, 8–1, 7 | 1, 8–3, 7 |
| | 5 | | | 6 | |
| 2, 12–3, 11 | 2, 10–3, 12 | 1, 10–2, 12 | 2, 11–3, 12 | 1, 11–2, 12 | 2, 12–3, 10 |
| 1, 12–3, 10 | 1, 10–3, 11 | 3, 12–1, 11 | 1, 11–3, 10 | 3, 12–1, 10 | 1, 12–3, 11 |
| 2, 10–1, 11 | 2, 11–1, 12 | 3, 10–2, 12 | 2, 10–1, 12 | 3, 11–2, 10 | 2, 11–1, 10 |
| | 7 | | | 8 | |
| 2, 15–3, 14 | 2, 13–3, 15 | 1, 13–2, 15 | 2, 14–3, 15 | 1, 14–2, 15 | 2, 15–3, 13 |
| 1, 15–3, 13 | 1, 13–3, 14 | 3, 15–1, 14 | 1, 14–3, 13 | 3, 15–1, 13 | 1, 15–3, 14 |
| 2, 13–1, 14 | 2, 14–1, 15 | 3, 13–2, 14 | 2, 13–1, 15 | 3, 14–2, 13 | 2, 14–1, 13 |
| | 9 | | | 10 | |
| 6, 7–5, 9 | 6, 8–5, 7 | 4, 8–6, 7 | 6, 9–5, 7 | 4, 8–6, 7 | 6, 7–5, 8 |
| 4, 7–5, 8 | 4, 8–5, 9 | 5, 7–4, 9 | 4, 9–5, 8 | 5, 7–4, 9 | 4, 7–5, 9 |
| 6, 8–4, 9 | 6, 9–4, 7 | 5, 9–6, 8 | 6, 8–4, 7 | 5, 8–6, 9 | 6, 9–4, 8 |
| | 11 | | | 12 | |
| 4, 13–6, 15 | 4, 14–6, 13 | 5, 14–4, 13 | 4, 15–6, 13 | 5, 15–4, 13 | 4, 13–6, 14 |
| 5, 13–6, 14 | 5, 14–6, 15 | 6, 13–5, 15 | 5, 15–6, 14 | 6, 13–5, 14 | 5, 13–6, 15 |
| 4, 14–5, 15 | 4, 15–5, 13 | 6, 14–4, 15 | 4, 14–5, 13 | 6, 15–4, 14 | 4, 15–5, 14 |
| | 13 | | | 14 | |
| 4, 10–6, 12 | 4, 11–6, 10 | 5, 11–4, 10 | 4, 12–6, 10 | 5, 12–4, 10 | 4, 10–6, 11 |
| 5, 10–6, 11 | 5, 11–6, 12 | 6, 10–5, 12 | 5, 12–6, 11 | 6, 10–5, 11 | 5, 10–6, 12 |
| 4, 11–5, 12 | 4, 12–5, 10 | 6, 11–4, 12 | 4, 11–5, 10 | 6, 12–4, 11 | 4, 12–5, 11 |
| | 15 | | | 16 | |
| 8, 14–9, 15 | 8, 13–9, 14 | 7, 13–8, 14 | 8, 15–9, 14 | 7, 15–8, 14 | 8, 14–9, 13 |
| 7, 14–9, 13 | 7, 13–9, 15 | 9, 14–7, 15 | 7, 15–9, 13 | 9, 14–7, 13 | 8, 15–7, 13 |
| 8, 13–7, 15 | 8, 15–7, 14 | 9, 13–8, 15 | 8, 13–7, 14 | 9. 15–8, 13 | 7, 14–9, 15 |
| | 17 | | | 18 | |
| 7, 10–8, 11 | 7, 12–8, 10 | 9, 12–7, 10 | 7, 11–8, 10 | 9, 11–7, 10 | 7, 10–8, 12 |
| 9, 10–8, 12 | 9, 12–8, 11 | 8, 10–9, 11 | 9, 11–8, 12 | 8, 10–9, 12 | 9, 10–8, 11 |
| 7, 12–9, 11 | 7, 11–9, 10 | 8, 12–7, 11 | 7, 12–9, 10 | 8, 11–7, 12 | 7, 11–9, 12 |

| | 19 | | | 20 | |
|---|---|---|---|---|---|
| 13, 11–14, 12 | 13, 10–14, 11 | 15, 10–13, 11 | 13, 12–14, 11 | 15, 12–13, 11 | 13, 11–14, 10 |
| 15, 11–14, 10 | 15, 10–14, 12 | 14, 11–15, 12 | 15, 12–14, 10 | 14, 11–15, 10 | 15, 11–14, 12 |
| 13, 10–15, 12 | 13, 12–15, 11 | 14, 10–13, 12 | 13, 10–15, 11 | 14, 12–13, 10 | 13, 12–15, 10 |

The twenty-five *v* lines are:

1, 2, 3,–4, 5, 6–7, 8, 9–,10, 11, 12–13, 14, 15–1, 11, 5–1, 4, 13
1, 8, 12–1, 9, 14–2, 6, 15–2, 11, 14–2, 10, 7–2, 9, 5–3, 12, 13
3, 7, 15–3, 4, 8–3, 6, 10–4, 10, 14–4, 7, 11–5, 7, 13–5, 12, 15
6, 8, 14–6, 9, 12–8, 11, 15–10, 13, 9

Finally we consider the notation: 2, 7, 10–1, 8, 12. Here 2, 7, 10 are the three *V* points in a *v* line and 1, 8, 12 are the three *V* points on another *v* line. But only two points determine a line, so we can get a number of notations to represent the same Pascal Point. We choose the three 7, 10–8, 12; 2, 7–1, 8; 2, 10–1, 12. These three notations represent the same Pascal Point. The reason is that the *V* point numbers on the five given lines run by threes: 1, 2, 3–4, 5, 6–7, 8, 9–10, 11, 12–13, 14, 15; and only two sets can be used as a conic. Any other combination possible as 2, 7,–8, 12 would use points from three lines; hence it would have no meaning in a Pascal group. If we pick from the sixty Pascal Lines given above the three which have the three notations above as Pascal Points, we have:

| 1 | 2 | 3 |
|---|---|---|
| 7, 10–8, 12 | 2, 7–1, 8 | 2, 10–1, 12 |
| 9, 10–8, 11 | 3, 7–1, 9 | 2, 11–3, 12 |
| 9, 12–7, 11 | 3, 8–2, 9 | 1, 11–3, 10 |

Now let us investigate the other six points on these three Pascal lines in the following way. Take point 9, 10–8, 11 and choose from the twenty-five *v* lines above the two, the first of which contains the points 9 and 10, and the second the points 8 and 11.

We get the notation, 9, 10–8, 11; 9, 13–8, 15; and 10, 13–11, 15 for the same Pascal Point, and, going back to the sixty Pascal Lines given above, we can choose three that contain these notations.

Continuing the investigation as far as necessary we find seven more lines:

| 4 | 5 | 6 | 7 |
|---|---|---|---|
| 7, 15–9, 14 | 3, 15–1, 14 | 3, 4–2, 5 | 4, 8–5, 9 |
| 7, 13–8, 14 | 1, 13–2, 15 | 1, 4–2, 6 | 6, 8–5, 7 |
| 8, 15–9, 13 | 3, 13–2, 14 | 1, 5–3, 6 | 6, 9–4, 7 |

| 8 | 9 | 10 |
|---|---|---|
| 11, 15–10, 13 | 5, 12–4, 10 | 5, 15–4, 14 |
| 12, 13–11, 14 | 6, 10–5, 11 | 4, 13–6, 15 |
| 15, 12–14, 10 | 6, 12–4, 11 | 5, 13–6, 14 |

The ten Pascal Lines above, one from each of the ten Pascal groups, form a closed set of ten lines and ten points related such that three lines go through each point and three points lie on each line. The same point lies in three groups and three points lie in each group. (See Figure 5.)

An interesting property of this last figure is that it is self-dual. If we start with five points and dualize the construction we would arrive at the same set of ten points and ten lines. I would like to name this set a geometric decagram.

All of these results can be constructed by linear constructions beginning with five lines which form a pentagon.

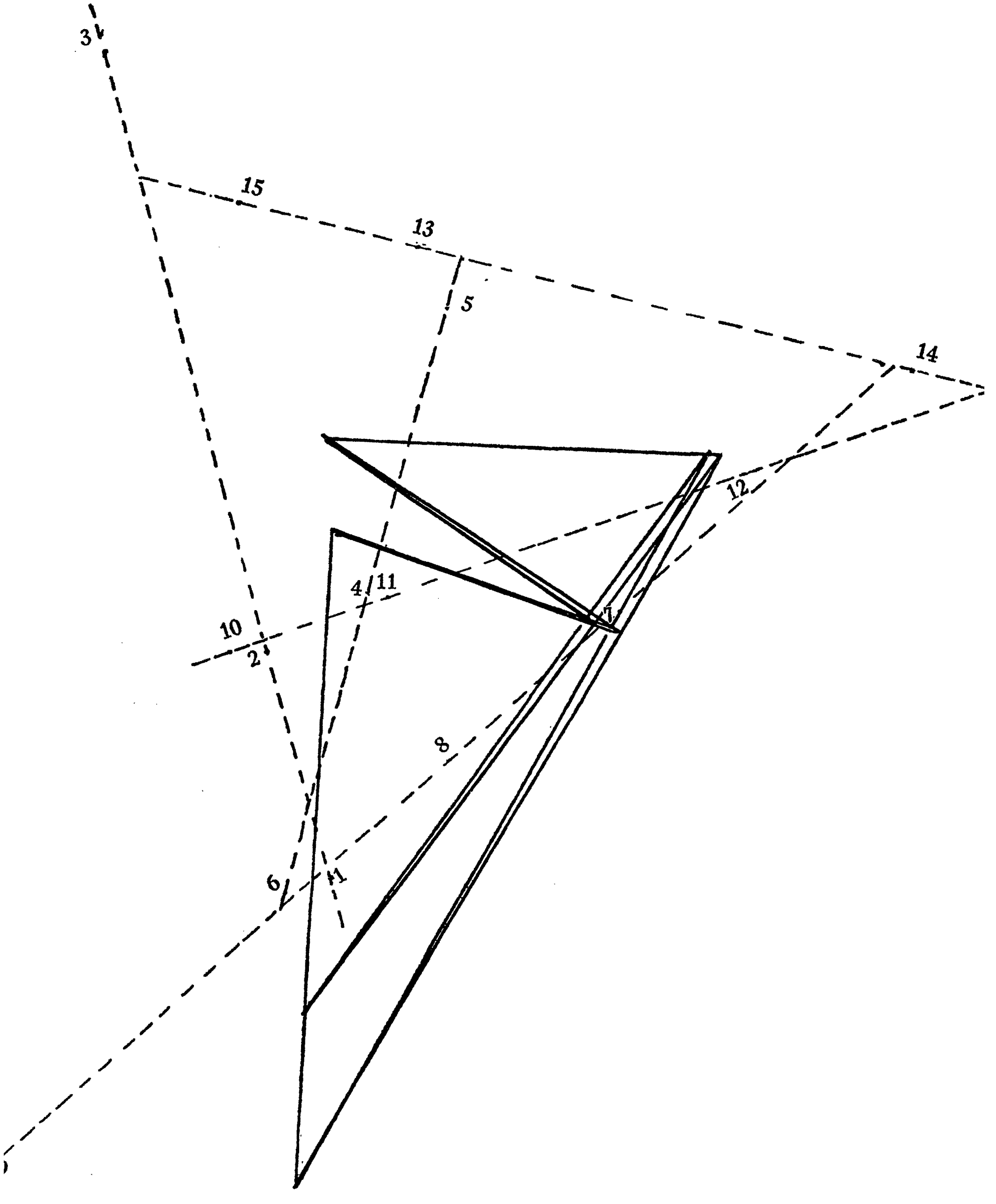


FIG. 5.

## ANSWERS

**A409.** Since $x^2 \equiv a \pmod{q}$ has no solutions in integers, $a^{(q-1)/2} \equiv -1 \pmod{q}$ by Euler's criterion, so that $a^{r(q-1)/2} \equiv -1 \pmod{q}$ where $r$ is odd.

Varying $r$ over $S$ and adding, we obtain the result. By taking $a=p$, $q=3$ and $S=\{1, 3, 5, \cdots 2n-1\}$ we have Problem 634, this MAGAZINE.

**A410.** Let $R=\{0, x_1, \cdots, x_n\}$. Then

$$\prod_{1\leq j\leq n} x_j^2 = x_p.$$

Clearly $x_p \neq 0$. But then

$$\left(\prod_{\substack{1\leq j\leq n\\ j\neq p}} x_j^2\right) x_p = 1$$

so that

$$\left(\prod_{\substack{1\leq j\leq n\\ j<p\\ j\neq i}} x_j^2\right) x_i x_j = x_i^{-1}, \quad 1 \leqq i \leqq n.$$

**A411.** Let $p_n$ and $p_{n+1}$ be successive odd primes. Since $p_n$ and $p_{n+1}$ are odd, $p_n+p_{n+1}$ is even and $p_n<(p_n+p_{n+1})/2<p_{n+1}$. Thus since $p_n$ and $p_{n+1}$ are successive primes, $(p_n+p_{n+1})/2$ is composite.

**(Quickies on page 170)**

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# ON THE PRODUCT OF NORMS OF ORTHOGONAL VECTORS

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In a right triangle $ABC$, where $C$ is the vertex of the right angle, let $CH$ be the altitude through $C$ (Figure 1). Let $r$ be the radius of the circumscribed circle

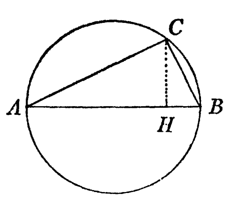


FIG. 1.

of the triangle. We observe that $(CA)(CB)=2r(CH)$. The vector proof of this equality is of some interest since the equality involves the product of the norms of two vectors and suggests relations involving products of norms of several orthogonal vectors in a unitary space. In this note we obtain a generalization of the proposition for a unitary space.

**1. Definitions and notations.** A unitary space will be denoted by $E$. The *norm* of a vector $\alpha$ will be $|\alpha|$. The *inner product* of two vectors $\alpha$ and $\beta$ is $(\alpha, \beta)$. Scalars will be denoted by small Latin letters.

2. THEOREM. *Suppose that $\alpha$ and $\beta$ are two nonzero orthogonal vectors in a real Euclidean plane. Take $\xi = a\alpha + b\beta$ with $a+b=1$ and such that $(\xi, \beta-\alpha)=0$ and let $\delta = h\alpha + k\beta$ satisfy $|\delta| = |\delta-\alpha| = |\delta-\beta|$. Then $|\alpha|\ |\beta| = 2|\xi|\ |\delta|$.*

*Proof.* We observe that $(\xi, \beta) = b|\beta|^2$, $(\xi, \alpha) = a|\alpha|^2$, and that $(\xi, \beta-\alpha)=0$ implies that $(\xi, \beta) = (\xi, \alpha)$. Thus $a|\alpha|^2 = b|\beta|^2$. It is clear that $|\xi|^2 = a^2|\alpha|^2 + b^2|\beta|^2 = a(a|\alpha|^2) + b(b|\beta|^2) = a|\alpha|^2(a+b) = a|\alpha|^2$. Hence we have

$$|\xi|^2 = a|\alpha|^2 = b|\beta|^2. \tag{1}$$

We see that $|\alpha+\beta|^2 = |\alpha|^2 + |\beta|^2$ which implies that

$$\begin{aligned} ab|\alpha+\beta|^2 &= ab|\alpha|^2 + ab|\beta|^2 = b[a|\alpha|^2] + a[b|\beta|^2] \\ &= (a+b)a|\alpha|^2 = a|\alpha|^2 = |\xi|^2. \end{aligned} \tag{2}$$

By (1) we can say that

$$a^2|\alpha|^4 = |\xi|^4. \tag{3}$$

But $a^2|\alpha|^4 = (a|\alpha|^2)(b|\beta|^2) = ab|\alpha|^2|\beta|^2$. Therefore, using (2), we can express (3) as $ab|\alpha|^2|\beta|^2 = |\xi|^2[ab|\alpha+\beta|^2]$ which implies

$$|\alpha|^2|\beta|^2 = |\xi|^2[|\alpha+\beta|^2] \quad \text{or} \quad |\alpha|\ |\beta| = |\xi|\ |\alpha+\beta|. \tag{4}$$

On the other hand, $\delta = h\alpha + k\beta$ and $|\delta|^2 = |\delta-\alpha|^2$ imply that $|\alpha|^2 = 2(\delta, \alpha) = 2h|\alpha|^2$. Thus $h = 1/2$. Similarly, $|\delta|^2 = |\delta-\beta|^2$ implies that $k=1/2$. Now we have $\delta = 1/2\alpha + 1/2\beta$, and $|\delta| = 1/2|\alpha+\beta|$. Therefore, we can express (4) as $|\alpha|\ |\beta| = 2|\xi|\ |\delta|$.

One may observe that this proof holds for the case of a complex two-dimensional unitary space if $a$, $b$, $h$, and $k$ are real numbers.

3. THEOREM. *Suppose $\{\alpha_1, \cdots, \alpha_k\}$, $k \geqq 2$, is a set of nonzero orthogonal vectors in $E$. Take $\xi = \sum_{i=1}^k a_i\alpha_i$ where $\sum_{i=1}^k a_i = 1$ and the $a_i$ are real for $i=1, \cdots, k$ such that $(\xi, \alpha_i - \alpha_j) = 0$ for $i, j = 1, \cdots, k$ and let $\delta = \sum_{i=1}^k h_i\alpha_i$, $h_i$ real for $i=1, \cdots, k$, satisfy $|\delta| = |\delta - \alpha_i|$ for $i = 1, \cdots, k$. Then $2|\delta|\ |\xi|^{k-1} \leqq (\prod_{i=1}^k |\alpha_i|)$. (Equality holds if and only if $k=2$.)*

*Proof.* By hypothesis, $(\xi, \alpha_i - \alpha_j) = 0$ which implies that $(\xi, \alpha_i) = (\xi, \alpha_j)$. Now we observe that $(\xi, \alpha_j) = a_j|\alpha_j|^2$, $j = 1, \cdots, k$. Therefore $a_1|\alpha_1|^2 = \cdots = a_k|\alpha_k|^2$.

Now we have

$$\begin{aligned} |\xi|^2 &= a_1^2|\alpha_1|^2 + \cdots + a_k^2|\alpha_k|^2 = a_1(a_1|\alpha_1|^2) + \cdots \\ &\quad + a_k(a_k|\alpha_k|^2) = a_j|\alpha_j|^2, \quad j = 1, \cdots, k. \end{aligned} \tag{5}$$

Since $\{\alpha_1, \cdots, \alpha_k\}$ is a set of nonzero orthogonal vectors and $\sum_{i=1}^k a_i = 1$, we have $0 < a_j < 1$ for $j = 1, \cdots, k$.

By [1] we know that $2\delta = \sum_{i=1}^k \alpha_i$. Thus

$$4|\delta|^2 = \left|\sum_{i=1}^{k}\alpha_i\right|^2 = \sum_{i=1}^{k}|\alpha_i|^2. \tag{6}$$

We observe that

$$\begin{aligned}\left(\prod_{i=1}^{k}a_i\right)(4|\delta|^2) &= \left(\prod_{i=1}^{k}a_i\right)\left(\sum_{i=1}^{k}|\alpha_i|^2\right)\\ &= (a_2\cdot\cdots\cdot a_k)(a_1|\alpha_1|^2)\\ &\quad + (a_1\cdot a_3\cdot\cdots\cdot a_k)(a_2|\alpha_2|^2) + \cdots\\ &\quad + (a_1\cdot\cdots\cdot a_{k-1})(a_k|\alpha_k|^2)\\ &= [(a_2\cdot\cdots\cdot a_k) + (a_1\cdot a_3\cdot\cdots\cdot a_k) + \cdots\\ &\quad + (a_1\cdot\cdots\cdot a_{k-1})]|\xi|^2.\end{aligned}$$

Since $a_i < 1$ for each $a_i$, $\prod a_i < 1$ for $i = 1, \cdots, k$. We easily see that $[(a_2\cdot\cdots\cdot a_k)+(a_1\cdot a_3\cdot\cdots\cdot a_k)+\cdots+(a_1\cdot\cdots\cdot a_{k-1})]<(a_1+\cdots+a_k)=1$. Thus

$$\left(\prod_{i=1}^{k}a_i\right)(4|\delta|^2) < |\xi|^2. \tag{7}$$

By (5), we see that $|\xi|^{2k}=(\prod_{i=1}^{k}a_i)(\prod_{i=1}^{k}|\alpha_i|^2)$. Thus

$$\prod_{i=1}^{k}a_i = \frac{|\xi|^{2k}}{\prod_{i=1}^{k}|\alpha_i|^2}. \tag{8}$$

Combining (7) and (8) we get

$$|\xi|^2 > 4|\delta|^2\left[\frac{|\xi|^{2k}}{\prod_{i=1}^{k}|\alpha_i|^2}\right] \quad \text{or} \quad \prod_{i=1}^{k}|\alpha_i|^2 > 4|\delta|^2|\xi|^{2(k-1)}$$

which implies

$$2|\delta|\,|\xi|^{k-1} < \prod_{i=1}^{k}|\alpha_i|, \quad k > 2.$$

It is clear that when $|\alpha_i| < 1$ for $i=1, \cdots, k$, we have a lower bound and an upper bound for $\prod|\alpha_i|$.

**Reference**

1. A. R. Amir-Moez, Vector solutions of geometric problems and their generalizations, this MAGAZINE, 38 (1965) 301.

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**ERRATA:** (September–October, 1966 issue, page 225, "A Remark on a Note of S. M. Shah"). Line 4 below the diagram should read "$\cdots -AP\cdot PB$, $\therefore AP^2$," instead of "$\cdots +AP\cdot PB$, $\therefore AP^2$."

# A PROBLEM IN 2-DIMENSIONAL HEAT FLOW

W. J. SCHNEIDER, Syracuse University

Let $D$ be a thin circular metal plate consisting of the points $|z| \leqq 1$ in the complex plane and let $C$ be the circumference of $D$. Let $\alpha$ be a closed proper subarc of $|z| = 1$ and let $\beta$ be a closed subarc of $\alpha$ containing neither of the end points of $\alpha$. Let $J(\subset D)$ be a closed Jordan domain such that $J \cap C = \alpha$ and let $K(\subset J)$ be a closed Jordan domain such that $K \cap C = \beta$. (See Figure 1.)

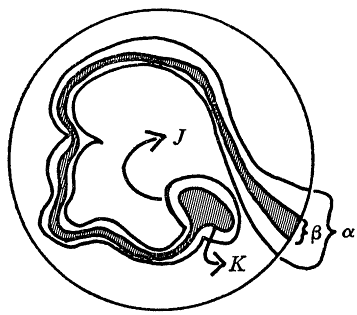


FIG. 1

PROBLEM. Given any $\epsilon > 0$ can we keep the points of $C$ at the following temperature $T(z)$:

$$T(z) = \begin{cases} \equiv 100 & (z \in \beta), \\ \equiv 0 & (z \in \text{complement of the interior of } \alpha \text{ on } C), \\ \text{continuous on} & (z \in \text{complement of the interior of } \beta \text{ on } C) \end{cases}$$

and have the steady state temperature $T(z)$ in the plate satisfy the following conditions:

(i) $|T(z) - 100| < \epsilon$ if $z \in K$,

(ii) $|T(z)| < \epsilon$ if $z \in (D - J)$?

Since $J$ and $K$ can "snake around" quite arbitrarily, this problem might well have practical applications to brain surgery. First we will show that this problem always has a solution and then discuss a number of drawbacks that might limit its practical application.

**Solution to the problem.** If $E$ is a compact set which does not separate the plane, then it follows from Mergelyan's Theorem [2] that if $f(z)$ is continuous on $E$ and analytic in the interior of $E$, we can approximate $f(z)$ uniformly on $E$ by polynomials. (Although the proof of Mergelyan's Theorem is neither elementary nor easy, its statement and many of its applications can be easily understood by anyone who has had an elementary course in complex variable.) Let $E$ be the union of $(D - J)$ and $K$ and $f(z)$ be the function which is identically 100 on $K$ and identically 0 on the closure of $(D - J)$. By Mergelyan's Theorem there exists a polynomial $P(z)$ such that $|f(z) - P(z)| < \epsilon/2$ on $E$. Let $S(z)$ be the real part of $P(z)$ and let $T(z)$ be any continuous function on $C$ which is identically 100 on $\beta$ and identically 0 on $C-\alpha$ and which differs from $S(z)$ on $C$ by less than $\epsilon/2$. Since $T(z)$ is continuous there exists a solution (call it $T(z)$) of the Dirichlet problem for the disk with boundary values $T(z)$. By the maximum and minimum principles for harmonic functions [1, p. 179], $T(z)$ satisfies (i) and (ii) and since it is harmonic inside the plate it is also the steady state solution of the heat flow problem.

**Remarks on practical applications.** Since the solution of the problem yields no bounds for $T(z)$ on $\beta-\alpha$, it is conceivable that such high or low temperatures might be required on $\beta-\alpha$ to remove the realization of the solution from the realm of practicality. This is also not a very efficient way of heating $K$ to be 100 and keeping $(D-K)$ at 0. Further, for the purpose of brain surgery, the solution of the three-dimensional analog to this problem would be required—if such a solution exists. Such a solution would appear to be much more difficult to obtain because, topologically, the number of possibilities of intertwining $K$ and $(D-K)$ in three dimensions as opposed to two dimensions is fantastically increased, and, the theory of complex variable is not applicable any longer. Also even in the case where we did have complex variable methods, we had to apply Mergelyan's Theorem, which is certainly one of the deepest and most powerful theorems in all of modern complex variable theory.

*Acknowledgment.* The author wishes to thank the referee for his many worthwhile suggestions and comments on the original manuscript.

**References**

1. L. V. Ahlfors, Complex Analysis, McGraw-Hill, New York, 1953.

2. S. N. Mergelyan, On the representation of functions by series of polynomials on closed sets, Amer. Math. Soc. Transl., 85 (1953).

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# CONSTRUCTIONS FOR CERTAIN CIRCLES OF CURVATURE AND RELATED EXTREME PROBLEMS

HERTA TAUSSIG FREITAG and ARTHUR H. FREITAG, Hollins College

We assume the familiar formulas for the radius of curvature for the graph of $f(x, y)=0$, namely,

$$R = \left| \frac{(1 + y'^2)^{3/2}}{y''} \right| \quad \text{for } y'' \neq 0 \tag{1a}$$

and

$$R = \left| \frac{(1 + x'^2)^{3/2}}{x''} \right| \quad \text{for } x'' \neq 0. \tag{1b}$$

These relationships may be translated into geometric language, thus enabling cally 100 on $\beta$ and identically 0 on $C-\alpha$ and which differs from $S(z)$ on $C$ by less than $\epsilon/2$. Since $T(z)$ is continuous there exists a solution (call it $T(z)$) of the Dirichlet problem for the disk with boundary values $T(z)$. By the maximum and minimum principles for harmonic functions [1, p. 179], $T(z)$ satisfies (i) and (ii) and since it is harmonic inside the plate it is also the steady state solution of the heat flow problem.

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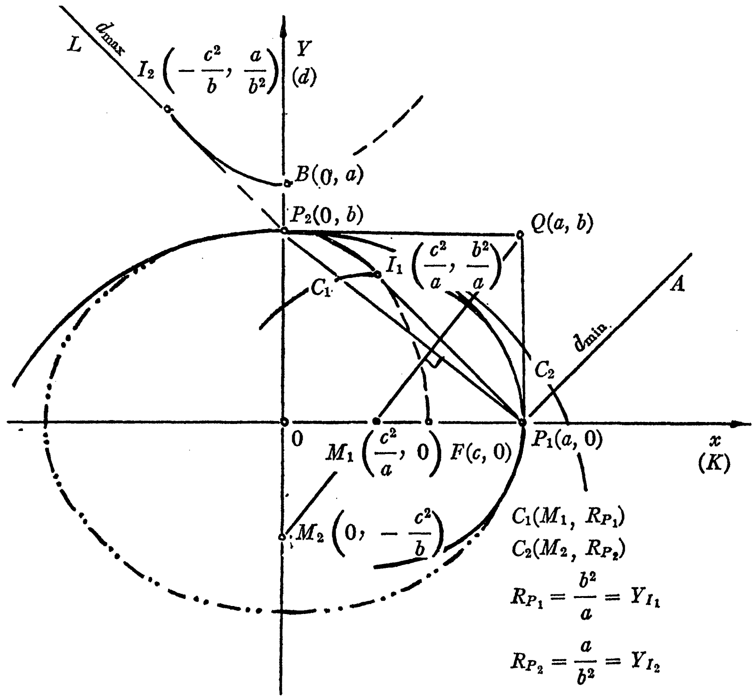


$d_{min}$ $\begin{cases} P_2I_1 \cdots \text{elliptical arc} \\ I_1P_1A \cdots \text{absolute value function} \end{cases}$

$d_{max}$ $\begin{cases} LI_2 \cdots \text{straight line} \\ I_2B \cdots \text{hyperbolic arc} \end{cases}$

FIG. 1.

the location of the centers of curvature at the vertices of conic sections. The resulting constructions are exceedingly simple—one of the mathematician's posited ingredients of elegance. Furthermore, those of us who are lacking in "artistic accomplishments" may be able to do better justice to the "curvaceousness" of the conics we draw if we can readily locate the circles of curvature at the vertices. Also, interrelations between analysis and Euclidean as well as analytic geometry become apparent. Finally, as a "bonus," we can utilize these results for certain extreme value problems, which—although simple—show rather unexpected relationships.

Without loss of generality we may assume the standard forms of the conics, adopting the usual symbolism. We will let the major axis of the ellipse, the transverse axis of the hyperbola, and the axis of the parabola be horizontal. Having the vertices $P_1(a, 0)$ and $P_2(0, b)$ apply to both the ellipse and the hyperbola, and the origin be the vertex of the parabola, we will label the corresponding centers of the circles of curvature $M_1$, $M_2$, and $M$, respectively. Formulas (1) lead to

$$(2) \qquad R_{p_1} = \frac{b^2}{a}, \qquad R_{p_2} = \frac{a^2}{b}, \qquad R_0 = 2p.$$

Hence, we obtain

$$M_1 = \left(a - \frac{b^2}{a}, 0\right) = \left(\frac{c^2}{a}, 0\right)$$

$$(3) \qquad M_2 = \left(0, b - \frac{a^2}{b}\right) = \left(0, \frac{-c^2}{b}\right)$$

$$M = (2p, 0).$$

We are now ready to formulate the constructions which lead to a location of these centers. For the ellipse (See Figure 1):

(1) Complete the rectangle on $a$ and $b$ to obtain $Q(a, b)$.

(2) Draw diagonal $P_1P_2$.

(3) Construct a perpendicular from $Q$ to $P_1P_2$.

(4) Intersect this perpendicular line with the $x$-axis to obtain $M_1$, and with the $y$-axis for $M_2$.

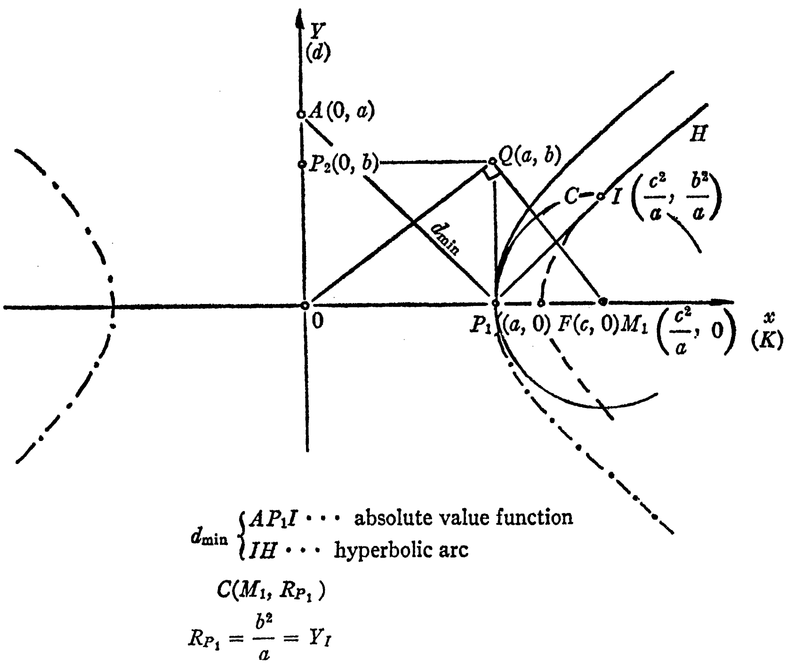


**Fig.** 2.

If the hyperbola is at hand (Figure 2), a slight variation will lead to $M_1$. Note that $M_1$ has the same coordinates for the hyperbola as for the ellipse. Instead of drawing diagonal $P_1P_2$ (step (2)), draw diagonal $OQ$. This means drawing a perpendicular to $OQ$ at $Q$ and intersecting it with the $x$-axis to locate $M_1$.

Finally, center $M$—for the parabola—should be found on the axis of this curve such that the focus becomes the midpoint of the line segment from $O$ to $M$ (Figure 3).

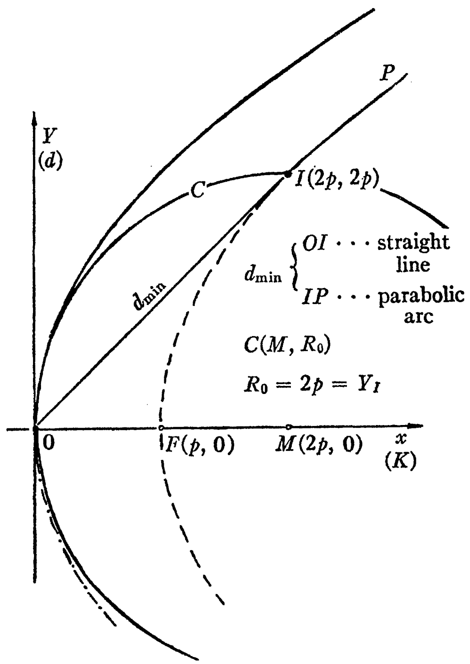


FIG. 3.

The proof of these constructions is readily made by analytic geometry. Since the slope $m_{P_1P_2} = -b/a$, the required perpendicular has equation $ax - by = a^2 - b^2$. This straight line intersects the coordinate axes at

$$\left(\frac{a^2 - b^2}{a}, 0\right) = \left(\frac{c^2}{a}, 0\right),$$

and at

$$\left(0, \frac{b^2 - a^2}{b}\right) = \left(0, -\frac{c^2}{b}\right),$$

respectively. Proceed similarly for the hyperbola. The case of the parabola is obvious.

Now let us stipulate a first quadrant elliptical arc by $b^2x^2 + a^2y^2 = a^2b^2$, such that the domain and range are restricted to $0 \leqq x \leqq a$, and $0 \leqq y \leqq b$, respectively. Let a point $K_x$ be symbolized by $K_x = (K, 0)$ for $0 \leqq K < \infty$. We will determine the minimum distance, $d_{\min}$, of the point $K_x$ from the elliptical arc.

We wish to minimize $\bar{d} = \sqrt{(x-K)^2 + y^2)}$ and obtain $x - K + yy' = 0$ to be correlated with $b^2x + a^2yy' = 0$. This leads to the coordinates of $D$, the nearest point on the arc, as:

$$x_D = \frac{a^2K}{c^2}, \qquad y_D^2 = \left(\frac{b}{c^2}\right)^2 [c^4 - (aK)^2].$$

Hence $-c^2/a \leqq K \leqq c^2/a$ or, with the above restrictions on $K$, $0 \leqq K \leqq c^2/a$. It is here that we associate this problem with the curvature condition for $P_1$ (formulas (2) and (3)) to obtain:

$$x_D = \begin{cases} \left(\frac{a}{c}\right)^2 K & \text{for } 0 \leqq K \leqq \frac{c^2}{a} \\ a & \text{for } \frac{c^2}{a} \leqq K < \infty. \end{cases}$$

Substituting these values into our minimum distance $d_{\min}$, and simplifying, we obtain:

$$(4) \qquad d_{\min} = \begin{cases} \frac{b}{c}\sqrt{(c^2 - K^2)} & \text{for } 0 \leqq K \leqq \frac{c^2}{a} \\ |a - K| & \text{for } \frac{c^2}{a} \leqq K < \infty. \end{cases}$$

This means (see Figure 1) that the graph of $d_{\min}$ is a standard form elliptical arc for $0 \leqq K \leqq c^2/a$. The initial point is $(0, b)$. One of the semiaxes (horizontal) is $c$, and the other semiaxis is $b$. For $c^2/a \leqq K < \infty$ the graph becomes a linear absolute value function such that the absolute value of its slope is one, and its point of nondifferentiability is $(a, 0)$. Obviously, $0 \leqq d_{\min} < \infty$.

To investigate the related problem for $K_y = (0, K)$, we let $-\infty < K \leqq 0$, and determine the maximum distance, $d_{\max}$, of $K_y$ from the elliptical arc. Reasoning analogously,

$$(5) \qquad d_{\max} = \begin{cases} b - K & \text{for } -\infty < K \leqq -\frac{c^2}{b} \\ \frac{a}{c}\sqrt{(c^2 + K^2)} & \text{for } -\frac{c^2}{b} \leqq K \leqq 0. \end{cases}$$

Formula (5) (see Figure 1) results in a straight line through $(0, b)$ with slope minus one for the first portion of the maximum distance function, and a standard form hyperbolic arc whose vertical semitransverse axis is $a$, and whose conjugate semiaxis is $c$ for the second portion. The hyperbolic arc terminates at $(0, a)$ and $a \leqq d_{\max} < \infty$.

Similar results may be obtained in an analogous manner for the remaining two conic sections. For the hyperbola, we have

$$(6) \qquad d_{\min} = \begin{cases} |a - K| & \text{for } 0 \leqq K \leqq \frac{c^2}{a} \\ \frac{b}{c}\sqrt{(K^2 - c^2)} & \text{for } \frac{c^2}{a} \leqq K < \infty. \end{cases}$$

Again (see Figure 2) a linear absolute value function and a standard form hyperbolic arc come into play. The transverse semiaxis (horizontal) is $c$ units long, the conjugate semiaxis $b$ units.

Finally, in the case of the parabola the minimum distance function becomes

$$d_{\min} = \begin{cases} K & \text{for } 0 \leqq K \leqq 2p \\ 2\sqrt{p(K-p)} & \text{for } 2p \leqq K < \infty. \end{cases} \tag{7}$$

This, of course, represents a combination of a straight line and a parabolic arc (see Figure 3). The latter has its vertex at $F$ (the focus of the given parabola) and its own focus at $M$. Its axis is horizontal and it is similar to the given parabola.

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## NOTES ON SEMIRINGS

LOH HOOI-TONG, University of Malaya, Kuala Lumpur

The notion of the algebraic system known as a semiring, though at times briefly mentioned in books of modern algebra, has never been systematically discussed. The object of this note is to show the richness and the importance of semirings through interesting examples which are drawn from various fields in mathematics.

A nonempty set $G$ with a binary operation "$\cdot$" is called a groupoid and is denoted by $\langle G, \cdot\rangle$. If the elements of $\langle G, \cdot\rangle$ also obey the associative law, then $\langle G, \cdot\rangle$ is called a semigroup. A semiring is a nonempty set $S$ with two binary operations "$+$" and "$\cdot$" such that

(i) $\langle S, +\rangle$ is a commutative semigroup.

(ii) The left and the right distributive laws hold; i.e., for each $a$, $b$, $c$ of $S$, we have

$$a\cdot(b+c) = a\cdot b + a\cdot c,$$
$$(a+b)\cdot c = a\cdot c + b\cdot c.$$

A semiring is denoted by $\langle S, +, \cdot\rangle$.

An element 0 of a semiring $\langle S, +, \cdot\rangle$ is called a zero element if and only if, for each $a$ of $S$, the following two conditions are met:

(i) $0+a=a$, and

(ii) $0\cdot a = a\cdot 0 = 0$.

A semiring may or may not have a zero element; if it has, it is unique; if it does not have a zero element we can always add one to it [**1**].

With the above definition, it is obvious that every ring is a semiring. On the other hand, the world around us abounds with semirings which are not rings.

*Example* 1. The set $P$ of all positive integers forms a semiring $\langle P, +, \cdot\rangle$ under the usual definitions of addition and multiplication of integers.

Again (see Figure 2) a linear absolute value function and a standard form hyperbolic arc come into play. The transverse semiaxis (horizontal) is $c$ units long, the conjugate semiaxis $b$ units.

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*Example* 1. The set $P$ of all positive integers forms a semiring $\langle P, +, \cdot\rangle$ under the usual definitions of addition and multiplication of integers.

*Example* 2. The set $K$ of all subsets of a set forms a semiring $\langle K, \cap, \cup\rangle$, where $\cap$ and $\cup$ are the operations of taking intersection and union of subsets of a set.

*Example* 3. The set $C$ of all cardinal numbers less than a fixed infinite cardinal number forms a semiring $\langle C, +, \cdot\rangle$ where $+$ and $\cdot$ have the usual meaning of addition and multiplication of cardinal numbers.

*Example* 4. The set $J$ of all ideals of an associative ring forms a semiring $\langle J, +, \cdot\rangle$ where $+$ and $\cdot$ have the usual meaning of addition and multiplication of ideals.

*Example* 5. The set $M$ of all $n\times n$ matrices with nonnegative entries forms a semiring $\langle M, +, \cdot\rangle$ where $+$ and $\cdot$ have the usual meaning of addition and multiplication of matrices.

Other more interesting examples of semirings are the following:

1. Let $\langle A, +\rangle$ be a commutative semigroup and let $H$ be the set of all endomorphisms of $\langle A, +\rangle$. Clearly the set $H$ is not empty and $H$ is a semiring with operations $\oplus$ and $\odot$ defined as follows: For any two endomorphisms $\theta_1$ and $\theta_2$ of $H$, and every $a$ of $A$, we define

$$a(\theta_1 \oplus \theta_2) = a\theta_1 + a\theta_2,$$

and

$$a(\theta_1 \odot \theta_2) = (a\theta_1)\theta_2.$$

We call $\langle H, \oplus, \odot\rangle$ the semiring of endomorphisms of $\langle A, +\rangle$.

2. Let $\langle S, +, \cdot\rangle$ be a semiring and let $\theta_1$ and $\theta_2$ be any two endomorphisms of $\langle S, +\rangle$. We leave the operation $+$ of $\langle S, +, \cdot\rangle$ unchanged, and redefine the operation $\cdot$ by

$$a \times b = a\theta_1\cdot b\theta_2$$

for any $a$, $b$ of $S$. Then $\langle S, +, \times\rangle$ is a semiring.

One interesting point about the semiring $\langle S, +, \times\rangle$ is that the endomorphisms of $\langle S, +\rangle$ may be quite different. For instance, let $S$ be the collection of all polynomial functions of a variable $x$; then $\langle S, +, \cdot\rangle$ is a semiring where $+$ and $\cdot$ have the usual meaning of addition and multiplication of polynomial functions. We may take $\theta_1$ to be the endomorphism of $\langle S, +\rangle$ such that $\theta_1$ maps any $p$ of $S$ into the derivative of $p$ with respect to $x$, and $\theta_2$ to be the endomorphism which maps $p$ into the integral of $p$ with respect to $x$.

Finally, we give a semiring obtained from the pseudo metrics on an arbitrary set. Let $E$ be a nonempty set and $d: E\times E\rightarrow R_+$ a mapping from the cartesian product of $E$ with itself into the set $R_+$ of nonnegative real numbers. The mapping $d$ is a metric on $E$ provided we have, for every $x$, $y$ of $E$,

(i) $d(x, x)=0$,
(ii) $d(x, y)>0$, if $x\neq y$,
(iii) $d(x, y)=d(y, x)$,

(iv) $d(x, y)+d(y, z) \geqq d(x, z)$.
A metric is called a pseudo metric if condition (ii) is not satisfied.

Now, let $S$ be the set of all pseudo metrics on $E$. Define an operation $\dotplus$ in $E$ as follows: For any pseudo metrics $d_1$ and $d_2$ of $S$, and any $(a, b)$ of $E \times E$, $d_1 \dotplus d_2 = d_1(a, b)+d_2(a, b)$. It is obvious that $d_1 \dotplus d_2$ is again a pseudo metric and that $\langle S, \dotplus \rangle$ is a commutative semigroup.

We next introduce a partially-ordered relation $\leqq$ in $S$. For any $d_1$, $d_2$ of $S$ we define $d_1 \leqq d_2$ if and only if $d_1(a, b) \leqq d_2(a, b)$, for every $(a, b)$ of $E \times E$. Then $\langle S, \leqq \rangle$ is a partially-ordered set. This partially-ordered set is also a semilattice with a smallest element $d_0$ defined by $d_0(a, b)=0$ for every $(a, b)$ of $E \times E$, and every two elements $d_1$, $d_2$ of $\langle S, \leqq \rangle$ have a least upper bound $d_1 \vee d_2$ given by

$$(d_1 \vee d_2)(a, b) = \max \{d_1(a, b), d_2(a, b)\} = d_1(a, b) \vee d_2(a, b).$$

Our semiring of pseudo metrics $\langle S, \dotplus, \vee \rangle$ is obtained by taking $S$ as the set, $\dotplus$ as the multiplication operation and $\vee$ as the addition operation.

The author would like to thank Professor H. H. Teh for valuable suggestion and discussion.

### Reference

1. H. T. Loh and H. H. Teh, Fundamentals of semirings, Nanyang University Mathematical Tract Series, (to appear).

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## HOMEOMORPHISMS ON FINITE SETS

HENRY SHARP, JR., Emory University

**1. Introduction.** In the summer of 1963, the author conducted a seminar in topology for a small group of talented secondary school students. In the course of studying limit point relations on a finite set, one of these students conjectured the result indicated in Theorem 1. A lively group discussion following the conjecture produced the proof given below. Though very simple in itself, this theorem is helpful in that it suggests the use of square incidence matrices in the study of topologies on finite sets.

THEOREM 1. *A topology on a finite set is identified by the closures of the singleton subsets.*

*Proof.* Let $S$ be a finite topological space and suppose that the closure of each singleton subset is known. We show that if $A = \{s_1, s_2, \cdots, s_k\}$ is any subset of $S$ and $X^-$ denotes the closure of $X$, then

$$A^- = \bigcup_{i=1}^{k} \{s_i\}^-.$$

Let $p \in \{s_i\}^-$ and assume $p \neq s_i$. Each open set containing $p$ contains $s_i$ and hence has nonempty intersection with $A$. So $p \in A^-$. Now let $p \in A^-$ and assume

(iv) $d(x, y)+d(y, z)\geqq d(x, z)$.
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$p \notin A$. Each open set containing $p$ has nonempty intersection with $A$. Let $U_1, U_2, \cdots, U_q$ be the family of open sets containing $p$. The intersection $U$ of members of this family is an open set containing $p$. Hence $U$ contains some point $s_j \in A$ and $p \in \{s_j\}^-$.

The closures of all subsets being known, the topology consists of the family of complements.

**2. Finite topologies.** The following remarks on closure will clarify the role played by Theorem 1 in the sequel. To each topology $\mathfrak{J}$ on $S$ there corresponds a closure operator defined on the power set of $S$ which satisfies the Kuratowski closure properties:

$$\varnothing^- = \varnothing, \qquad M \subset M^-, \qquad (M^-)^- = M^-, \quad \text{and} \quad (M \cup N)^- = M^- \cup N^-.$$

Theorem 1 says that every member of $\mathfrak{J}$ can be identified once the restriction of the closure operator to singleton subsets is known. On the other hand, to produce a topology the assignment of closures cannot be arbitrary. If the set $S$ is finite, then the equation $A^- = \bigcup \{s_i\}^-$ follows immediately from the property $(M \cup N)^- = M^- \cup N^-$. Hence any assignment of closures to the singleton subsets of $S$ determines a topology on $S$ if for each $s_i \in S$,

$$s_i \in \{s_i\}^- \quad \text{and} \quad (\{s_i\}^-)^- = \{s_i\}^-. \tag{1}$$

Throughout the remainder of this paper $S$ represents a finite set of $n$ elements labeled $s_1, s_2, \cdots, s_n$. Given a topology $\mathfrak{J}$ on $S$, $T$ is the *incidence matrix* for $\mathfrak{J}$ iff

$$T = [t_{ij}], \quad i, j = 1, 2, \cdots, n \text{ and for each } i \text{ and } j$$

$$t_{ij} = \begin{cases} 1 & \text{if } s_j \in \{s_i\}^-, \\ 0 & \text{otherwise.} \end{cases}$$

For example, on a set of three elements the incidence matrices for the discrete and indiscrete topologies are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

If $A$ denotes any subset of the given set $S$, then the *characteristic function* corresponding to $A$ is that function

$$f: S \to \{0, 1\}$$

for which

$$f(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \notin A. \end{cases}$$

There is a one-to-one correspondence between the subsets of $S$ and the set of functions $f: S \to \{0, 1\}$.

For each $i$ let $F_i$ be the set corresponding to the characteristic function

$\{(s_1, t_{i1}), (s_2, t_{i2}), \cdots, (s_n, t_{in})\}$; and for each $j$ let $B_j$ be the set corresponding to the characteristic function $\{(s_1, t_{1j}, (s_2, t_{2j}), \cdots, (s_n, t_{nj})\}$. By definition, $F_i = \{s_i\}^-$; thus it is the minimal closed set containing $s_i$. It is easily shown that $B_j$ is the minimal open set containing $s_j$. Thus a minimal basis for $\mathfrak{T}$ consists of the empty set, $\emptyset$, together with the distinct sets $B_j$.

THEOREM 2. *The zero-one square matrix $T$ is an incidence matrix for a topology iff for all $i$ $t_{ii}=1$ and for all $i \neq j \neq k$ if $t_{ij}=t_{jk}=1$ then $t_{ik}=1$.*

*Proof* (Sufficiency). We show that conditions (1) hold. If $t_{ii}=1$ then $s_i \in \{s_i\}^-$ and $(\{s_i\}^-)^- \supset \{s_i\}^-$. The second hypothesis implies that if $s_j \in \{s_i\}^-$ then $\{s_j\}^- \subset \{s_i\}^-$. From this $(\{s_i\}^-)^- \subset \{s_i\}^-$, and $T$ is an incidence matrix for a topology.

(Necessity). If $T$ is the incidence matrix for a topology $\mathfrak{T}$ then the Kuratowski closure properties hold, and conditions (1) imply that $t_{ii}=1$ and that if $s_j \in \{s_i\}^-$ then $\{s_j\}^- \subset \{s_i\}^-$.

Recently a proof of this theorem has been published by V. Krishnamurthy [2]. The two proofs differ widely.

**3. Transformations.** A *relation*, $\alpha$, in $S$ is a set of ordered pairs in $S \times S$; and with $\alpha$ we associate an $n \times n$ zero-one *matrix* $A = [a_{ij}]$ in which

$$a_{ij} = \begin{cases} 1 & \text{if } (s_i, s_j) \in \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The *transpose*, $[a_{ji}]$, of $A = [a_{ij}]$ is denoted by $A'$. If $A$ is the matrix of a relation $\alpha$, then $A'$ is the matrix of the relation $\alpha^{-1}$. We assume hereafter that all relation matrices are of fixed order $n$, and that the underlying set is $S$.

The *product* of two relation matrices $A$ and $B$ is defined (as usual) by

$$AB = C = [c_{ij}] = \left[ \sum_k a_{ik} b_{kj} \right];$$

however the arithmetic is Boolean:

| + | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| · | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

If we agree that $0<1$, an *order* between matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ is defined by

$$A \leqq B \text{ iff } a_{ij} \leqq b_{ij} \quad \text{for each } i \text{ and } j;$$

and $A<B$ means, in addition, that $a_{ij}<b_{ij}$ for some $i$ and $j$. Denoting the *identity* matrix by $I$, we agree that $A$ is *reflexive* iff $A \geqq I$. If $A$ is reflexive, then $A^2 \geqq A$. It is well known [**1**, p. 209] that $A$ is *transitive* iff $A^2 \leqq A$; hence a reflexive relation is transitive iff $A^2 = A$. The relation matrix $A$ is *symmetric* iff $A' = A$.

From Theorem 2 we conclude immediately that a reflexive zero-one square matrix $T$ is the incidence matrix for a topology on $S$ iff $T^2 = T$.

Any relation $\alpha$ in $S$ may be considered a transformation (possibly many-valued) on a subset of $S$ into $S$. If $A$ denotes the matrix corresponding to $\alpha$, then $\alpha$ is a function if there is at most one "1" in each row, and $\alpha$ is on $S$ into $S$ if there is at least one "1" in each row.

THEOREM 3. *If $A$ is the matrix corresponding to the relation $\alpha$, then $\alpha$ is*
(1) *a function iff $A'A \leqq I$;*
(2) *on $S$ iff $AA' \geqq I$;*
(3) *a function on $S$ onto $S$ iff $A'A = I$.*

*Proof.* $A'A = [\sum a_{ki}a_{kj}]$ and $AA' = [\sum a_{ik}a_{jk}]$.

(1) $\alpha$ is a function iff for each $k$ the condition $a_{ki} = a_{kj} = 1$ implies that $i = j$. Thus $\alpha$ is a function iff $\sum a_{ki}a_{kj} = 0$ for $i \neq j$.

(2) For each $i$, $\sum a_{ik}a_{ik} = 1$ iff there is a $k$ such that $a_{ik} = 1$. Thus $\sum a_{ik}a_{ik} = 1$ for all $i$ iff the domain of $\alpha$ is $S$.

(3) If $\alpha$ is a function, then $A'A \leqq I$. If $\alpha$ is onto, then for each $j$ there is a $k$ such that $a_{kj} = 1$. Hence $\sum a_{kj}a_{kj} = 1$, and $A'A = I$. On the other hand, if $A'A = I$ then $\alpha$ is a function and (reversing the argument just given) $\alpha$ is onto. Since the domain is finite, domain $\alpha$ = range $\alpha = S$ and, in fact, $\alpha$ is one-to-one.

COROLLARY. *If $A'A = I$, then $AA' = I$.*

**4. Continuity.** We assume now that $\alpha$ is a function on $S$ into $S$, and that $A$ is its corresponding matrix. Thus $A'A \leqq I$ and $AA' \geqq I$.

If $V$ is a subset of $S$ corresponding to the characteristic function (Section 2) $\{(s_1, v_1), \cdots, (s_n, v_n)\}$, then the column vector

$$A'\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \cdot & & \\ \cdot & & \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

characterizes the image set $\alpha(V)$. If $\mathfrak{J}$ is a topology on $S$ with matrix $T$, then the $j$th column in $A'T$ characterizes the image, $\alpha(B_j)$, of the minimal open set in $\mathfrak{J}$ containing $s_j$. Furthermore, the $j$th column in $(A'T)A = A'TA$ characterizes the union of all image sets $\alpha(B_i)$ such that $(s_i, s_j) \in \alpha$.

The preceding argument establishes the following theorem.

THEOREM 4. *Let $\mathfrak{J}$ and $\mathfrak{J}^*$ be topologies on $S$. Then the function $\alpha$:$(S, \mathfrak{J}) \rightarrow (S, \mathfrak{J}^*)$ is continuous (on $S$) iff $A'TA \leqq T^*$.*

COROLLARY. *The function $\alpha$: $(S, \mathfrak{J}) \rightarrow (S, \mathfrak{J})$ is continuous (on $S$) iff $TA \leqq AT$.*

If $\alpha$ is one-to-one, then $\alpha^{-1}$: $(S, \mathfrak{J}^*) \rightarrow (S, \mathfrak{J})$ is a function with corresponding matrix $A'$. By Theorem 4, $\alpha^{-1}$ is continuous iff $AT^*A' \leqq T$. But then $T^* \leqq A'TA$, which proves the following theorem.

THEOREM 5. *Let $\mathfrak{J}$ and $\mathfrak{J}^*$ be topologies on $S$. The one-to-one function $\alpha$: $(S, \mathfrak{J}) \rightarrow (S, \mathfrak{J}^*)$ is a homeomorphism iff $A'TA = T^*$.*

COROLLARY. *The one-to-one function* $\alpha$: $(S, \mathfrak{J})\rightarrow(S, \mathfrak{J})$ *is a homeomorphism iff* $TA = AT$; *that is, iff the permutation matrix* $A$ *commutes with* $T$.

The seminar referred to in the first paragraph was supported by NSF grant GE 802 to Emory University.

**References**

1. Garrett Birkhoff, Lattice Theory (rev. ed.), Amer. Math. Soc., New York, 1948.
2. V. Krishnamurthy, On the number of topologies on a finite set, Amer. Math. Monthly, 73 (1966) 154.

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## BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

*Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.*

*One Hundred Problems in Elementary Mathematics.* By Hugo Steinhaus. Basic Books, Inc., New York, 1964. 174 pp. $4.95.

An elementary problem means here, a problem requiring in its solution only high school mathematics. Otherwise these problems require a creative mind and as such they demonstrate the beauty and the originality of mathematics. I wish that a mathematics teacher would forget sometimes the normal flow of day to day teaching and give his class a taste of the best in mathematics. It is not just learning a new recipe and then training its application by doing problems which are fundamentally alike. Here are problems which can serve for such demonstration. They do not fit any particular chapter. They have to be "carried" around for a few days before the right technique is finally found. If the teacher strings out the solution of such a problem for a week or two, devoting to it daily maybe 10–15 minutes, he can teach the student the drama of the solution of a nonroutine problem and the beauty of creative thinking. It creates motivation for many gifted children and calls to mathematics those precious few who have the intuition, the gift, and the passion for abstract exploration.

The problems are mostly surprisingly new and different from those found in the usual puzzle sections. Most of the problems have solutions given in the second part of the book which present a kaleidoscopic brilliant diversity. He who chooses to read the solutions instead of attempting to find them himself will get familiar with an astonishing variety of mathematical techniques. Sometimes the solution takes a totally unexpected turn. Reading that (a) everybody in the group was unacquainted with six others (b) everybody belonged to some mutually acquainted triplet (c) there was no group of 4 persons in which all members knew one another (d) there was no group of 4 persons in which nobody knew anybody (e) everybody belonged to a triplet not known to one another (f) everybody could find among the persons whom he did not know a person with

COROLLARY. *The one-to-one function* $\alpha: (S, \mathfrak{J}) \rightarrow (S, \mathfrak{J})$ *is a homeomorphism iff* $TA = AT$; *that is, iff the permutation matrix* $A$ *commutes with* $T$.

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There are problems on numbers which naturally extend the usual material taught in high school algebra on divisibility, irrationality, sequences, on polygons, circles and ellipses, on polyhedra and spheres.

This little book can be a true friend, if tucked away into your library and used at odd times as a special treat, to help you brush off the harassing routine of everyday from your shoulders.

FRANTISEK WOLF, University of California, Berkeley

*Elementary Analysis*. By Richard McCoart, Malcom Oliphant, and Anne Scheerer. Holden-Day, San Francisco, 1964. xi+251 pp. $7.95.

This excellent text in the calculus of one argument is well suited for a one year course in calculus for a highly qualified group of high school seniors or for a one year terminal course in calculus at the college level. With suitable omissions, it may be used as the text for the first semester of a three term sequence of rigorous calculus courses. The choice of material for this book is fine, the material is well motivated and presented in a crisp and rigorous style, and the problem sets are quite appealing.

The first chapter deals with the elementary operations of set theory, mathematical induction, and inequalities. The treatment of induction is highlighted by several nice examples and thought provoking exercises.

The second chapter deals with the definition of the function, several special functions (such as the linear, quadratic, and step functions), and a few topics from analytic geometry. The function is defined in terms of ordered pairs and the authors remain consistent in their functional notation throughout the text. The chapter on limits and continuity presents the usual material, as well as rather full treatments of one-sided limits and asymptotes.

The chapter on derivatives develops the usual formulas and applies the derivative concept to solving maximum-minimum problems and to curve sketching. (The authors omit a discussion of concavity.) The mean value theorem is developed carefully and its usefulness as a means for numerical approximations is pointed out in the exercises. The integral is discussed first for step functions, and for a continuous function, $f$, it is defined to be the supremum of the integrals of those step functions less than or equal to $f$ on the interval. The integral is applied to finding areas and volumes of revolution. The final chapter deals with the exponential and logarithmic functions. (The trigonometric functions and infinite series are not treated in this book.)

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S. M. ROBINSON, Union College

*The Geometry of Incidence*. By Harold L. Dorwart. Prentice-Hall, Englewood Cliffs, New Jersey, 1966. xvii+156 pp. $6.25.

In common with a group of mathematicians the author deplores the fact that "higher geometry has lost ground as a subject for study." *The Geometry of Incidence* was written to "revive an interest in geometry—specifically in projective geometry." The book goes a long way in the right direction and deserves to succeed in its main objective.

The style is informal, divorced from pedantic formalism, and altogether charming. It should appeal to the audience at which it is aimed—high school seniors, persons in teacher training courses in mathematics, and scientists with an interest in mathematics. Though the author does not list professional mathematicians as a portion of his intended audience he should not be astonished to find them receptive.

The first chapter deals with Klein's classification of geometrics and a justification for the title of the book. The notion of *ideal* elements is introduced in a very natural way, and the beauties of duality are well illustrated.

The real projective plane is defined in Chapter II and Euclidean models are discussed. The notions of a *plane configuration* and a *configuration* or *incidence table* (of basic importance in the rest of the book) are carefully considered. The question of "realizability" of an incidence table paves the way for some classical theorems of plane projective geometry in Chapter III.

The final chapter is concerned with finite projective planes and the intriguing combinatorial questions regarding their existence. The interrelation between perfect difference sets, complete sets of Latin squares, and finite projective planes is explored. The chapter touches on the work of many modern mathematicians and brings the reader to the frontier of knowledge in this area of combinatorics. Thus it exists as an admirable complement to the first three chapters where the results are classical and the uninitiated reader may get the idea that all has been accomplished.

In an otherwise laudatory collection of comments it must be stated that some of the figures are confusing; for example, Figures 63 and 74. In fact, in the latter, only 5 of a total of 13 lines are correctly identified in relation to the incidence table on p. 135. (By evil mischance it is reproduced on the jacket.) These are, however, rather minor objections to an otherwise attractively written and attractively printed little volume.

J. W. T. YOUNGS, University of California, Santa Cruz is to lead the student through a carefully developed sequence of propositions so that he may gain an appreciation of the art of mathematics." Used with a class of students with strong mathematical aptitude and motivation, this book provides an excellent vehicle with which to achieve both of these aims. The presentation, however, is not at all geared to a mediocre group.

S. M. ROBINSON, Union College

*The Geometry of Incidence.* By Harold L. Dorwart. Prentice-Hall, Englewood Cliffs, New Jersey, 1966. xvii+156 pp. $6.25.

In common with a group of mathematicians the author deplores the fact that "higher geometry has lost ground as a subject for study." *The Geometry of Incidence* was written to "revive an interest in geometry—specifically in projective geometry." The book goes a long way in the right direction and deserves to succeed in its main objective.

The style is informal, divorced from pedantic formalism, and altogether charming. It should appeal to the audience at which it is aimed—high school seniors, persons in teacher training courses in mathematics, and scientists with an interest in mathematics. Though the author does not list professional mathematicians as a portion of his intended audience he should not be astonished to find them receptive.

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J. W. T. YOUNGS, University of California, Santa Cruz

*Calculus with Analytic Geometry*. By Albert G. Fadell. Van Nostrand, Princeton, N. J., 1964. xxvii+705 pp. $9.75.

This lucidly written and superbly illustrated text in the calculus of one variable merits serious consideration for adoption with a class of well prepared and promising students. It is also an excellent choice for summer institute courses for high school teachers. This reviewer has been particularly impressed by the motivating discussions preceding the introduction of new concepts, the juxtaposition of an informal explanation of the content of a theorem with the formal statement and the rich collection of examples and exercises.

Some other features of the book are an excellent discussion of functions and sequences, a discussion of proximity which precedes the chapter on limits and prepares the student for the epsilon-delta arguments he will meet, and a classification (along with clarifying illustrations) of types of discontinuities. The definition of the integral is given in terms of upper and lower sums for a sequence of partitions and the integrability theorem is first proved for the class of monotone functions. After a careful discussion of uniform continuity, the integrability of continuous functions is established.

This book contains the usual material on derivatives, integrals, transcendental functions, and analytic geometry, but treats this material unusually well. The concluding chapters contain a good introduction to vectors and matrices, followed by a discussion of analytic geometry and polar coordinates.

There seem to be a few occasions where the discussion becomes unduly abstruse. In particular, the discussion of inductive sets and the introduction of cluster points into the discussion of limits seem likely to cause some difficulty for students. However, these instances are quite rare and certainly do not dim this reviewer's enthusiasm for the book.

Professor Fadell's book is written at a level of sophistication comparable to that attained in the well known texts of Apostol and Johnson-Kiokemeister, and along with its second volume, it should provide formidable competition for these classics.

S. M. ROBINSON, Union College

**Word Problems**

If cars leave Carson City at the minute
That westbound jets are overhead Des Plaines,
And buses from Columbus
Are blanketing the compass
Can you get out of Norwalk on a train?

If alcohol's added to water,
And adequate stirring maintained,
How long does an olive
Require to dissolve if
The mixture is frequently drained?

MARLOW SHOLANDER

## BRIEF MENTION

*Algebra and the Elementary Functions.* By Bevan K. Youse. Dickenson, Belmont, Calif., 1966. ix+297 pp. $7.95.

In addition to the usual material found in integrated algebra and trigonometry texts, an elementary treatment of other precalculus topics such as analytic geometry, linear equations and determinants, and vector algebra is included.

*College Algebra.* By Merlin M. Ohmer and Clayton V. Aucoin. Blaisdell, Waltham, Mass., 1966. xii+320 pp. $7.50.

Intended for terminal students, precalculus students, and prospective high school teachers. Designed as a high-level review of the standard two-year high school algebra course, with emphasis on those topics essential to analytic geometry and calculus.

*Limits: A Transition to Calculus.* By O. Lexton Buchanan, Jr., Houghton Mifflin, Boston, 1966. iv+186 pp. $2.20 (paper).

A leisurely treatment of limits and sequences with attractive illustrations which terminate with a chapter on infinite series. Applications include rate of change, area, volume and work problems.

*Techniques of Differentiation and Integration: A Program for Self-instruction.* By Herman Meyer and Robert V. Mendenhall. McGraw-Hill, New York, 1966. x+168 pp. $3.50 (paper), $5.95 (hardback).

Intended as a supplement to a teacher and textbook, its purpose is to "remove the preoccupation with the techniques of differentiation and integration from the classroom."

*A Collection of Problems in Analytic Geometry.* By D. V. Kletenik. Pergamon, Oxford, 1966. Part I: Analytic Geometry in the Plane; ix+186 pp. $2.95 (paper). Part II: Three-dimensional Analytic Geometry; ix+137 pp. $2.95 (paper).

These two volumes in the Commonwealth and International Library series were translated from the Russian by E. R. Dawson and F. J. Rayner. Part I contains 718 problems, Part II 543 problems. Summaries of relevant theory, hints, examples, and answers are provided. Vector algebra and determinants are included.

*The Contributions of Faraday and Maxwell to Electrical Science.* By R. A. R. Tricker. Pergamon, Oxford, 1966. ix+289 pp. $4.50 (paper).

The first part is a commentary which includes interesting biographical sketches of these great physicists; the second part consists of selected papers.

*Elementary Vector Algebra, 2nd ed.* By A. M. Macbeath. Oxford, London, 1966. 137 pp. $2.00.

A geometrical, nonaxiomatic approach. Answers and hints on solutions to the exercises have been added.

*Introduction to Elementary Vector Analysis.* By J. C. Tallack. Cambridge, London, 1966. ix+140 pp. $3.50.

An elementary introduction to the algebra of vectors and applications of vectors to geometry and mechanics. Differentiation, integration, and the scalar product are also covered.

*An Introduction to Matrices, Vectors, and Linear Programming.* By Hugh G. Campbell. Appleton-Century-Crofts, New York, 1965. xiv+244 pp. $6.50.

Assumes previous knowledge of elementary algebra and plane geometry. More elementary than most texts on matrix algebra; contains material on mathematical systems, convex sets, and the simplex method.

*Matrices and Transformations.* By Anthony J. Pettofrezzo. Prentice-Hall, Englewood Cliffs, N. J., 1966. vii+133 pp. $3.95.

Consists of four chapters: Matrices, Inverses and Systems of Matrices, Transformations of the Plane, Eigenvalues and Eigenvectors. Intended to familiarize the reader with the role of matrices in abstract algebraic systems and to illustrate their effective use as a mathematical tool in geometry.

*Fortran IV Programming and Computing.* By James T. Golden. Prentice-Hall, Englewood Cliffs, N. J., 270 pp. $4.95 (paper), $6.60 (hard cover).

A college-level introduction to computing and programming. Designed to develop ability to generate algorithms and to create strategies for problem solving on a digital computer. May be used for self-study or to supplement a numerical methods course. Includes sections on matrix and Boolean algebra and Monte Carlo techniques.

*Dictionary/Outline of Basic Statistics.* By John E. Freund and Frank J. Williams. McGraw-Hill, New York, 1966. vii+195 pp. $2.95 (paper), $5.95 (hardcover).

Part I is a dictionary of over 1000 statistical terms and Part II is an outline of statistical formulas. Material chosen reflects some of the recent developments in the areas of decision theory, experimental design, the theory of games, computer techniques, operations research, and statistical inference.

*Elementary Statistics, 2nd ed.* By Paul G. Hoel. Wiley, New York, 1966. ix+351 pp. $7.25.

This edition features a broader coverage and a geometric approach to probability, increased discussion and emphasis on sampling distributions, new illustrative exercises, and specially marked supplementary treatments of new topics such as Bayes' Formula and nonlinear regression.

*An Introduction to Probability and Statistics.* By Beryle Hume. Univ. of Western Australia Press, Nedlands, 1966. xv+286 pp. $3.75.

This very attractive elementary text contains numerous examples and exercises. Based on an experimental text, *Introductory Probability and Statistical Inference*, prepared for the Commission on Mathematics, C.E.E.B., New York.

*Trends in Elementary School Mathematics.* By Lloyd Scott. Rand McNally, Chicago, 1966. ix+215 pp. $3.50 (paper).

An overview of the modern elementary school mathematics program with particular attention to the practical interests of teachers. An extensive list of references is included.

*Basic Concepts of Elementary Mathematics, 2nd ed.* By William L. Schaaf. Wiley, New York, 1965. xix+384 pp. $6.95.

This edition contains new or more material on operations, mappings, truth tables, number theory, modular arithmetic, inequalities, geometry, permutations, combinations, and probability.

*Numbers and Arithmetic.* By John N. Fujii. Blaisdell, New York, 1965. xi+559 pp. $8.50.

Designed as a prealgebra review course.

*Modern Mathematics for the Elementary Teacher.* By Leslie A. Dwight. Holt, Rinehart, and Winston, New York, 1966. ix+598 pp. $8.95.

Includes structures and development of number systems and an introduction to algebraic and geometric concepts and methods. Content and method are combined.

*Guidebook to Departments in the Mathematical Sciences in the U. S. and Canada*, 2nd ed. Raoul Hailpern, Editor. The MAA, SUNY at Buffalo, Buffalo, N. Y. 14214, 1966. 80 pp. $.50 (paper).

This revised and enlarged printing of the original 1965 edition contains about 1200 entries. Location, size, staff, library facilities, course offerings, and special features of departments in mathematical sciences are presented in summary form. This useful booklet is a project of the MAA Committee on Advisement and Personnel.

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## PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

*Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.*

### PROPOSALS

**656.** *Proposed by Sidney Kravitz, Dover, New Jersey.*

The editor of a mathematics journal said to his assistant, "I have here a cryptarithm which shows a two digit number being multiplied by itself. You will note that the subproducts are not shown, only the number being squared and the final product."

The assistant said, "I've tried to solve this cryptarithm but the solution is not unique. If you told me whether the number being squared were odd or even I might be able to give you a solution."

Afraid of being overheard, the editor whispers the answer to his assistant. The assistant said, "I was hoping you'd say that. I now know the solution."

Unfortunately, a spy for a rival mathematical journal had planted a "mike" in the room. Although he has not seen the cryptarithm, he has overheard the entire conversation. He is able to solve the cryptarithm. Can you?

**657.** *Proposed by C. Stanley Ogilvy, Hamilton College, New York*

Ship $A$ is anchored 9 miles out from a point $O$ on a straight shoreline. Ship $B$ is anchored 3 miles out opposite a point 6 miles from $O$. A boat is to proceed

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Ship $A$ is anchored 9 miles out from a point $O$ on a straight shoreline. Ship $B$ is anchored 3 miles out opposite a point 6 miles from $O$. A boat is to proceed from $A$ to some point on the shore, pick up a passenger, and take him to ship $B$. It costs the boat owner $1 per mile to run his boat, whether there is a passenger aboard or not. Where should the owner contract to pick up the passenger so that his net profit (from $A$ to shore to $B$) shall be a maximum? We can assume that the passenger insists on a straight line course from the pickup point to $B$.

**658.** *Proposed by Kaidy Tan, Fukien Normal College, Fukien, China.*

Construct a cyclic quadrilateral so that each side touches one of four fixed circles.

**659.** *Proposed by C. J. Mozzochi, University of Connecticut.*

Let $(X, \sigma)$ be a measurable space with an uncountable number of measurable sets. Let $(Y, \tau)$ be a topological space with an uncountable number of open sets. Let $f$ and $g$ be measurable functions from $X$ into $Y$. Prove or disprove,

$$E = \{x \mid f(x) = g(x)\} \text{ is measurable.}$$

**660.** *Proposed by L. J. Upton, Port Credit, Ontario, Canada.*

Four lines in a plane are concurrent at $O$. The angles between the lines are each 45°. A circle is superimposed on this configuration so that $O$ lies within the circle. (a) Show that the alternate sectors cover one-half of the circle. (b) Show this result without use of the calculus.

**661.** *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Find all differentiable functions satisfying the functional equation

$$f(xy) = yf(x) + xf(y).$$

**662.** *Proposed by M. B. McNeil, University of Bristol, England.*

It is well known that between two real zeroes of a polynomial with real coefficients there is located one real zero of its derivative. Consider the more general question: Given the zeroes of a polynomial with possibly complex coefficients, what can be said about the zeroes of its derivative?

## SOLUTIONS

### Late Solutions

*Colin R. J. Singleton, Weybridge, Surrey, England:* **621, 622, 626, 627**; *Jerry Waxman, Brooklyn, New York:* **621, 622**; *E. L. Magnuson and G. C. Dodds (jointly), H. R.B.—Singer, Inc., State College, Pennsylvania:* **628**; *C. R. J. Singleton, Weybridge, Surrey, England; Stephen Spindler, Purdue University:* **629**; *J. R. Kuttler and Nathan Rubinstein, Johns Hopkins University; E. L. Magnuson, H.R.B.—Singer, Inc., State College, Pennsylvania; C. R. J. Singleton, Weybridge, Surrey, England; Daniel R. Stark, Cleveland State University:* **631**; *Alexandru Lupas, Institutul de Calcul, Cluj, Rumania; Stanley Rabinowitz, Far Rockaway, New York:* **632**; *Robert Abel, Gustavus Adolphus College, Minnesota; E. L. Magnuson, H.R.B.—Singer, Inc., State College, Pennsylvania; C. R. J. Singleton, Weybridge, Surrey, England; D. R. Stark, Cleveland State University:* **633**; *Gerald C. Dodds, H.R.B.—Singer, Inc., State College, Pennsylvania; Sandra Gossum, University of Tennessee, Martin Branch; G. E. Lewer, University of Sydney, Australia; E. L. Magnuson, H.R.B.—Singer, Inc., State College, Pennsylvania; C. R. J. Singleton, Weybridge, Surrey, England; Stephen Spindler, Purdue University:* **634.**

**Dissection of a Dodecahedron**

**635.** [November, 1966] *Proposed by P. D. and R. L. Goodstein, University of Leicester, England.*

Show that there is a closed path along the edges of a regular dodecahedron which divides the dodecahedron into two congruent parts each of which contains a pair of opposite faces of the dodecahedron.

*Solution by William Wernick, City College of New York.*

Suppose that the dodecahedron is placed at the origin with a pair of opposite vertices, $A$ and $B$, symmetrically placed on the upper and lower $z$-axis. Then any downward path from $A$ to $B$ is the point symmetric image of a corresponding upward path from $B$ to $A$. Together these determine a closed path that satisfies the requirements.

*Also solved by Michael Goldberg, Washington, D. C.; Colin R. J. Singleton, Weybridge, Surrey, England; Stanley Rabinowitz, Far Rockaway, New York; and the proposers.*

**Greatest Divisors of Even Integers**

**636.** [November, 1966] *Proposed by Vassili Daiev, Sea Cliff, New York*

The greatest divisors of the form $2^k$ of the numbers of the sequence 2, 4, 6, 8, 10, 12, 14, $\cdots$, are 2, $2^2$, 2, $2^3$, 2, $2^2$, 2, $\cdots$. Find the $n$th term of this sequence.

**I.** *Solution by Michael Goldberg, Washington, D.C.*

Express the numbers of the sequence 2, 4, 6, 8, $\cdots$, in the binary notation. Then $k$, the exponent of the greatest factor of the form $2^k$ is the number of zeroes at the right end of the binary representation.

**II.** *Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

If $a_n$ denotes the $n$th term, then it follows immediately that

$$\begin{aligned} a_{2n+1} &= 2^1, \\ a_{4n+2} &= 2^2, \\ a_{8n+4} &= 2^3, \\ \vdots\ & \quad \vdots \end{aligned}$$

In general,

$$a_{2^r m+2^{r-1}} = 2^r.$$

Note that every number $n$ can be expressed uniquely in the form $2^r m+2^{r-1}$.

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; Crayton W. Bedford, Phillips Academy, Andover, Massachusetts; Mickey Dargitz, Ferris State College, Michigan; Herta T.*

*Freitag, Hollins, Virginia; J. F. Leetch, Bowling Green State University, Ohio; Michael W. O'Donnell, University of Missouri; J. R. Purdy, Northeast Missouri State College; Stanley Rabinowitz, Far Rockaway, New York; David L. Silverman, Hughes Aircraft Company, El Segundo, California; Stephen Spindler, Purdue University; and the proposer.*

A variety of different expressions for the $n$th term of the sequence were given by the solvers.

**Equal Symmedians**

**637.** [November, 1966] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Prove that a triangle is isosceles if and only if it has two equal symmedians.

*Solution by Leon Bankoff, Los Angeles, California.*

Applying Stewart's Theorem to triangle $ABC$, in which $k_a$, the symmedian issuing from $A$, divides side $a$ into the segments $ac^2/(b^2+c^2)$ and $ab^2/(b^2+c^2)$, we obtain

$$k_a = bc\sqrt{2(b^2 + c^2) - a^2}/(b^2 + c^2).$$

Similarly,

$$k_b = ac\sqrt{2(a^2 + c^2) - b^2}/(a^2 + c^2).$$

The right sides of these equations are equal to each other if and only if $k_a = k_b$. Since $a$ and $b$ can be interchanged without affecting the equality, it follows that triangle $ABC$ is isosceles, with $AC = BC$.

*Also solved by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Colin R. J. Singleton, Weybridge, Surrey, England; Sister M. Stephanie Sloyan, Georgian Court College, New Jersey; Dmitrios Valthis, Halcis, Greece; Gregory Wulczyn, Bucknell University; and the proposer.*

Merrill Barnebey found the problem in *College Geometry*, by N. A. Court, Page 234, Problem 4.

**The Digits of a Factorial**

**638.** [November, 1966] *Proposed by Charles W. Trigg, San Diego, California.*

For what values of $n$ does the expanded form of $n!$ have exactly $2n$ digits?

*Solution by Brother Alfred Brousseau, Saint Mary's College, California.*

Stirling's formula in base 10 reads:

$$\log_{10} n! = (n + 1/2) \log_{10} n + .43429n + (1/2) \log_{10} 2\pi .43429[r(n)/12n]$$

where $1/(12n+1) < r(n)/12n < 1/12n$.

By trying a few values of $n$, it was found that $n$ is approximately 270 when $\log n!$ is approximately $2n$.

Further calculations with adjacent values of $n$ yielded the solutions: $n = 268, 267, 266$.

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; Donald Batman, MIT Lincoln Laboratory; Michael Goldberg, Washington, D. C.; Lawrence Kreicher, Lakewood, Ohio; Richard Laatsch and John McCarty (jointly), Miami University, Ohio; David L. Silverman, Hughes Aircraft Company, El Segundo, California; and Raymond E. Whitney, Lock Haven State College, Pennsylvania.* One incorrect solution was received.

**A Convex Quadrilateral Inequality**

**639.** [November, 1966] *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let $ABCD$ be a convex quadrangle and $P$ be the intersection of diagonals $AC$ and $BD$. Let the distance of $P$ from the sides $AB=a$, $BC=b$, $CD=c$, $DA=d$ be $x$, $y$, $z$, and $t$ respectively. Prove that

$$x+y+z+t<\tfrac{3}{4}(a+b+c+d).$$

*Solution by Leon Bankoff, Los Angeles, California.*

Let the bisectors of the angles between the diagonals $AC$ and $BD$ meet $AB$, $BC$, $CD$, $DA$ in $R$, $S$, $T$, $U$.

By a corollary of the Erdös-Mordell Theorem,

$$2(PS+PT)<PB+PC+PD$$
$$2(PT+PU)<PC+PD+PA$$
$$2(PU+PR)<PD+PA+PB$$
$$2(PR+PS)<PA+PB+PC$$

or $4(PR+PS+PT+PU)<3(PA+PB+PC+PD)$.

This inequality is stronger than the one proposed because

$$x+y+z+t\leqq PR+PS+PT+PU$$

and $PA+PB+PC+PD<a+b+c+d$.

*Also solved by Leon Bankoff, Los Angeles, California (second solution); Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Stanley Rubinowitz, Far Rockaway, New York and the proposer.*

**Tangent Circles**

**640.** [November, 1966] *Proposed by Leon Bankoff, Los Angeles, California.*

The semicircle $(O)$, described internally on the side $BC$ of a square $ABCD$, cuts the quadrant $BD$ of a circle with center at $A$ at a point $T$. Show that the circle $(P)$, tangent internally to the arc $BC$ at $T$ and tangent to the line $BC$ at $Q$ is also tangent to the circle on the diameter $AB$.

**I.** *Solution by Joseph D. E. Konhauser, University of Minnesota.*

Select axes and units so that the $x$-axis contains $BC$, so that the $y$-axis passes through the center of the square, and so that $C$ has coordinates (1, 0). Then $T$ has coordinates (3/5, 4/5), the circle $(P)$ has center (4/9, 1/3) and radius 4/9. The distance from the midpoint of $AB$ to (4/9, 1/3) is $13/9=1+4/9$. Therefore, circle $(P)$ of radius 4/9 is tangent to the circle of radius 1 on diameter $AB$.

**II.** *Solution by the proposer.*

With $AD$ as the radius of the circle of inversion and $A$ as center, invert the figure.

Circles $(O)$ and $(P)$ are self-inverse since they are cut orthogonally by the circle of inversion.

Lines $AD$ and $AB$ are self-inverse since they pass through the center of inversion.

The line $BC$ inverts into the circle on diameter $AB$.

Since the circle $(P)$ is tangent to line $BC$ it is also tangent to the circle on diameter $AB$.

*Also solved by Merrill Barnebey, Wisconsin State University, La Crosse; Crayton W. Bedford, Phillips Academy, Massachusetts; Joseph Bohac, St. Louis, Missouri; Brother Alfred Brousseau, St. Mary's College, California; Herta T. Freitag, Hollins College, Virginia (two solutions); Michael Goldberg, Washington, D. C.; Bruce W. King, Burnt Hills-Ballston Lake High School, New York; Lawrence Kreicher, Lakewood, Ohio; C. Stanley Ogilvy, Hamilton College, New York; C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; Sister M. Stephanie Sloyan, Georgian Court College, New Jersey; William Wernick, City College of New York; and the proposer (second solution).*

### A Convex Curve Property

**641.** [November, 1966] *Proposed by Yasser Dakkah, S. S. Boys' School, Qalqilya, Jordan.*

Prove that if

$$\sum_{i=1}^{n} x_i = S$$

and $0 < x_i$ $(i=1, 2, \cdots, n)$, then

$$\sum_{i=1}^{n} \cosh x_i \geqq n \cosh (S/n).$$

**I.** *Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

The result follows immediately from the well known inequality for convex functions, i.e.,

If $\phi(x)$ is convex, then

$$\frac{\phi(x_1) + \phi(x_2) + \cdots + \phi(x_n)}{n} \geqq \phi\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right).$$

Since cos $hx$ is everywhere convex ($D^2$ cos $hx > 0$), just replace $\phi(x)$ by cos $hx$ to give the desired result. Note that there is no necessity for the restriction $x_i > 0$.

**II.** *Solution by Douglas Lind, Charlottesville, Virginia.*

It is familiar that if $a_1, \cdots, a_n$ are $n$ positive real numbers, then

$$(a_1 + \cdots + a_n)/n \geqq (a_1 \cdots a_n)^{1/n}.$$

Putting $a_k = e^{xk} > 0$, we have

$$(1) \qquad e^{x_1} + \cdots + e^{x_n} \geqq ne^{S/n}.$$

Putting $a_k = e^{-x_k} > 0$ gives

$$e^{-x_1} + \cdots + e^{-x_n} \geqq ne^{-S/n}. \tag{2}$$

Adding (1) to (2) and dividing by 2 gives the result.

*Also solved by Donald Batman, MIT Lincoln Laboratory; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Michael Goldberg, Washington, D. C.; Joseph D. E. Konhauser, University of Minnesota; Stephen Spindler, Purdue University; and the proposer.*

**Comment on Problem 608**

**608.** [January, 1966] *Proposed by A. A. Gioia and A. M. Vaidya, Texas Technological College.*

Call a positive integer $n$ semiperfect if the sum of all the square free divisors of $n$ is $2n$. Prove that 6 is the only semiperfect number.

*Comment by A. M. Vaidya, Gujarat University, Ahmedabad, India.*

At the end of the solution to Problem 608 in the September, 1966, issue of the *Mathematics Magazine*, you have noted that "this problem appeared as Problem E 1755 in the *American Mathematical Monthly*, February, 1966, Page 203."

Since I had a hand in proposing both the problems 608 and E 1755, I would like to point out that the two problems are not identical. E 1755 asks for a proof that 6 is the only square-free perfect number and 608 asserts that 6 is the only semiperfect number. There is nothing in the definition of a semiperfect number which would make it obvious that a semiperfect number must be a square-free perfect number. That it is so, is what 608 proves.

To sum up, Problem 608 is stronger and more general than E 1755 in the sense that a solution to the former would constitute a solution to the latter but not vice versa.

**Comment on Problem 614**

**614.** [March and November, 1966] *Proposed by Charles W. Trigg, San Diego, California.*

In the cryptarithm

$$V\,E\,X\,I\,N\,G = M\,A\,T\,H,$$

the $X$ doubles as a multiplication sign and each other letter uniquely represents a positive digit. $M\,A\,T\,H$ is a permutation of consecutive digits. Find the two numerical solutions.

*Comment by the proposer.*

Each of the solutions published in the November, 1966, issue was obtained by exhaustion.

In the second solution, statements I made in discussing the trend of problem proposals in the *American Mathematical Monthly* problem departments (The Otto Dunkel Memorial Problem Book, 1957, Page 5) were quoted. Namely: "Problems whose difficulty lies principally in laborious computations have gradually disappeared. . . . Problems best solved by table searching are prescribed." The solvers suggest that these statements be used as criteria for problem proposals as well as for solutions. Many times a proposer does not have a solution to his problem or hopes that someone will provide a solution more elegant than his own laborious computations have provided. The editor cannot anticipate that no better solution will be submitted.

The late Norman Anning used to contend that with some numerical problems the most elegant approach was a rapid table search to find the solution, thus leaving time for more challenging intellectual activity. Of course, success with table or computer searching (the latter involving time for programming) does not preclude a rapid solution by other methods. The following solution provides a case in point.

Since the sum of the values of the letters is 45, we have $VE+ING+MATH\equiv 0$ (mod 9). Also, $(VE)(ING)\equiv MATH$ (mod 9). Consequently, $VE+ING+(VE)(ING)\equiv 0$ (mod 9). Hence we seek pairs of digits such that their sum plus their product is a multiple of 9. There are but four. It follows that $(VE)(ING)\equiv(4)(1)$ or $(7)(7)$ and $MATH$ is a permutation of 4567, or $(VE)(ING)\equiv(3)(6)$ or $(0)(0)$ and $MATH$ is a permutation of 3456.

If $MATH$ is a permutation of 4567, then $VE$ and $ING$ must be constructed from 1, 2, 3, 8, 9, that is, from the groups:

| | | | |
|---|---|---|---|
| $VE$ | 1, 3 | 1, 9 | 2, 8 |
| $ING$ | 2, 8, 9 | 2, 3, 8 | 1, 3, 9 |

If $MATH$ is a permutation of 3456, then $VE$ and $ING$ must be constructed from 1, 2, 7, 8, 9, that is, from the groups:

| | | | | |
|---|---|---|---|---|
| $VE$ | 1, 2 | 1, 8 | 2, 7 | 7, 8 |
| $ING$ | 7, 8, 9 | 2, 7, 9 | 1, 8, 9 | 1, 2, 9 |

Each one of these groups can be arranged in (2)(6) ways. But neither $E$ nor $G$ can be 1, so only 48 multiplications need be tested. Of these, 33 are immediately ruled out since they would give five-digit products, and in 13 of the others duplicate digits show up before the multiplication is complete. Finally, the solution is:

$$(18)(297) = 5346 = (27)(198).$$

Thus there is only one $MATH$, but two ways in which it can be $VEXING$.

If the restriction on the digits of $MATH$ be removed, there is another dual solution

$$(12)(483) = 5796 = (42)(138).$$

**Comment on Problem 615**

**615.** [March and November, 1966] *Proposed by Joseph Hammer, University of Sydney, Australia.*

Prove that in a three-dimensional convex surface whose volume is greater than the surface area numerically, infinitely many plane cross-sections can be found of which each area is greater than its perimeter.

*Comment by G. D. Chakerian, University of California at Davis.*

The published solution to Problem 615 is wrong, since in general $S \neq \int P(z)dz$. That "solution" contains, however, the kernel of a correct proof of a "better" result:

THEOREM. *Let $K$ be a 3-dimensional convex body with surface area $S < V$ = volume of $K$. Then there exist infinitely many cross-sections with perimeter $P < \pi/4(A)$.*

*Proof.* For each direction $u$, let

$$S(u) = \int_{-\infty}^{\infty} P(z, u)dz,$$

where $P(z, u)$ is the perimeter of the cross-section of $K$ by a plane orthogonal to direction $u$. Then $\int S(u)du = \pi^2 S$, where the integration is over the unit sphere [H. Hadwiger, Altes and Neues über Konvexe Körper, 1955, Page 96]. Hence if

$$P(z, u) \geqq \frac{\pi}{4} A(z, u)$$

for all $z$ and $u$, then $S \geqq V$. Hence $P < \pi/4(A)$ for some cross-section—hence for infinitely many ("nearby").

*A similar comment was submitted by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

## QUICKIES

***From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.***

**Q409.** Let $q$ be an odd prime and $a$ an integer such that $x^2 \equiv a \pmod{q}$ has no solutions in $x$. If $S$ is a set of $n$ odd positive integers then

$$\sum_{r \in s} a^{r(q-1)/2} + n \equiv 0 \pmod{q}$$

[*Submitted by P. N. Bajaj*]

**Q410.** A finite commutative ring with an identity and no zero divisors is a field.

[*Submitted by C. J. Mozzochi*]

**Q411.** Show that the sum of two successive primes is the product of at least three (not necessarily distinct) prime factors.

[*Submitted by John D. Baum*]

**(Answers on page 141)**

## ANSWERS

**A409.** Since $x^2 \equiv a \pmod q$ has no solutions in integers, $a^{(q-1)/2} \equiv -1 \pmod q$ by Euler's criterion, so that $a^{r(q-1)/2} \equiv -1 \pmod q$ where $r$ is odd.

Varying $r$ over $S$ and adding, we obtain the result. By taking $a=p$, $q=3$ and $S=\{1, 3, 5, \cdots 2n-1\}$ we have Problem 634, this MAGAZINE.

**A410.** Let $R=\{0, x_1, \cdots, x_n\}$. Then

$$\prod_{1 \leq j \leq n} x_j^2 = x_p.$$

Clearly $x_p \neq 0$. But then

$$\left(\prod_{\substack{1 \leq j \leq n \\ j \neq p}} x_j^2\right) x_p = 1$$

so that

$$\left(\prod_{\substack{1 \leq j \leq n \\ j < p \\ j \neq i}} x_j^2\right) x_i x_j = x_i^{-1}, \quad 1 \leqq i \leqq n.$$

**A411.** Let $p_n$ and $p_{n+1}$ be successive odd primes. Since $p_n$ and $p_{n+1}$ are odd, $p_n + p_{n+1}$ is even and $p_n < (p_n + p_{n+1})/2 < p_{n+1}$. Thus since $p_n$ and $p_{n+1}$ are successive primes, $(p_n + p_{n+1})/2$ is composite.

**(Quickies on page 170)**

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# ON THE PRODUCT OF NORMS OF ORTHOGONAL VECTORS

BEVERLY M. HYDE, Texas Technological College

In a right triangle $ABC$, where $C$ is the vertex of the right angle, let $CH$ be the altitude through $C$ (Figure 1). Let $r$ be the radius of the circumscribed circle

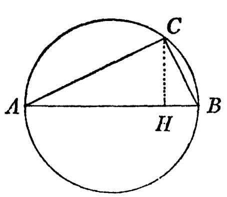


FIG. 1.

of the triangle. We observe that $(CA)(CB) = 2r(CH)$. The vector proof of this equality is of some interest since the equality involves the product of the norms of two vectors and suggests relations involving products of norms of several orthogonal vectors in a unitary space. In this note we obtain a generalization of the proposition for a unitary space.

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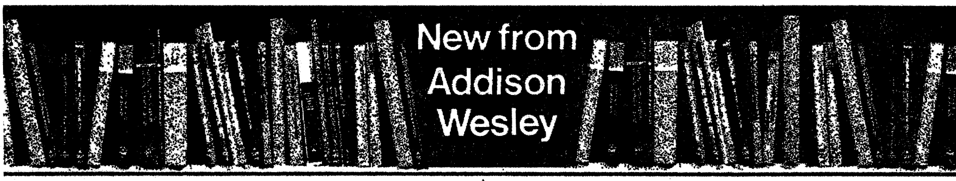


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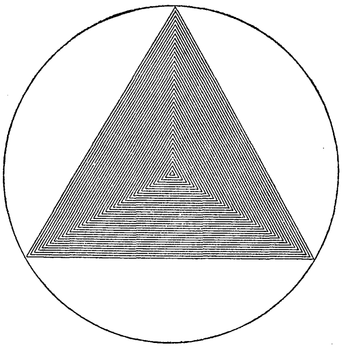

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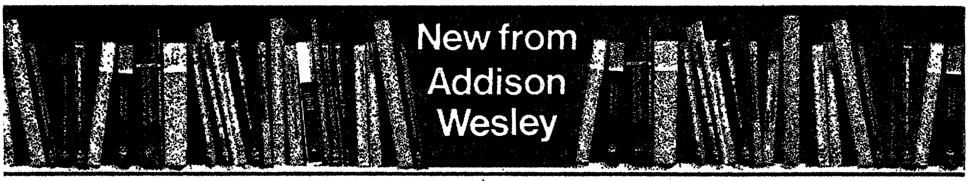


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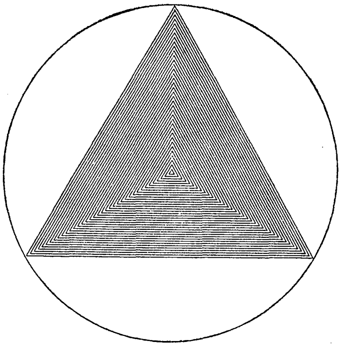

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